COMP 330 Fall 2023
Assignment 6
Due date: 5^{th} December 2023

Posted on November 17^{th} 2023

This is the sixth of 6 assignments. There are **6** questions for credit. The homework is due on `CrowdMark` **at 11:59pm**.

Important 1: Solutions have to be in **pdf** format. Every question must be on a **separate** pdf and submitted through Crowdmark. We prefer solutions prepared on LaTeX but that is **not required**. We will **not** accept jpg photos of handwritten solutions. We will not accept Word documents. Either of these formats can be exported to pdf. **TAs will remove marks for blurry/illegible submissions.**

Important 2: *All* of the questions on this assignment are *excellent preparation* for the final exam.

Note: Unless otherwise specified, assume that a Turing machine refers to a semi-infinite tape Turing machine like the one described in class.

Question 1. [10 points] Let $\Sigma = \{0, 1\}$. Consider the following *partial* function $f : \Sigma^* \rightharpoonup \Sigma^*$

$$f(v) = \begin{cases} 1 & \text{if } v = \langle M, w \rangle \text{ where } M \text{ is a TM}, w \in \Sigma^*, \text{ and } M \text{ loops on } w \\ \text{undefined} & \text{otherwise} \end{cases}$$

Show that f is not computable.

Question 2. [30 points] ***Read this carefully.*** Let $\Sigma = \{0, 1\}$ and consider the following languages

1. $L_{\texttt{EXACTLY\_ONE}} = \{\langle M, x \rangle : M \text{ is a TM}, x \in \Sigma^*, L(M) = \{x\}\}$
2. $L_{\texttt{AT\_MOST}} = \{\langle M, x \rangle : M \text{ is a TM}, x \in \Sigma^+, |L(M)| \leq \texttt{dec}(x)\}$

   Note: $\texttt{dec}(x)$ converts a binary string to its corresponding decimal number. For example, $\texttt{dec}(0110) = 6$.
3. $L_{\texttt{NOT\_EQ}} = \{\langle M_1, M_2 \rangle : M_1, M_2 \text{ are TMs}, L(M_1) \neq L(M_2)\}$

For each language, determine whether it is

1. Decidable.

2. Undecidable, but CE.
3. Undecidable, but co-CE.
4. Neither CE nor co-CE.

If you answer 1., give a decision procedure. If you answer 2. or 3., give a mapping reduction and a procedure. If you answer 4., give two mapping reductions. If you give a procedure, a proof of correctness is not required. If you give a mapping reduction, you may only reduce from the following problems (and their complements): $\texttt{HP}, \texttt{AP}, \texttt{ALL}, \texttt{EMPTY}$. **No marks will be awarded if you reduce from another problem or if you use Rice's theorem.**

Question 3. [10 points] Let Σ be a non-empty alphabet and let F be a finite family of CE languages which partitions Σ^*. Show that each language in F must be decidable.

Question 4. [10 points] Let $\Sigma = \{0, 1\}$ and consider the following language

$$L_{\texttt{LOOP}} = \{\langle M \rangle : M \text{ is a TM}, M \text{ loops on at least one input string } x \in \Sigma^*\}$$

Show that $L_{\texttt{LOOP}}$ is neither CE nor co-CE by showing that $\texttt{HP} \leq_m \texttt{LOOP}$ and $\overline{\texttt{HP}} \leq_m \texttt{LOOP}$.

Question 5. [20 points] Let $\Sigma = \{0, 1\}$ and consider the following languages

1. $L_{\texttt{BOUNDED}} = \{\langle M, x \rangle : M \text{ is a TM}, x \in \Sigma^*, M$ never moves its pointer more than $|x|$ tape symbols away from the left endmarker $\vdash\}$
2. $L_{\texttt{DEC}} = \{\langle M \rangle : M \text{ is a TM}, L(M) \text{ is decidable}\}$

For each language, determine whether it is decidable or undecidable. If it is decidable, give a decision procedure which decides it. If it is undecidable, you *must* use Rice's theorem, if applicable, or a mapping reduction, otherwise. [Hint: Given some input string x, you may assume that the number of possible instantaneous configurations of a TM where the pointer never moves more than $|x|$ tape symbols away from the left endmarker is at most $|\Gamma|^{|x|} \cdot \binom{|x|+1}{1} \cdot \binom{|Q|}{1}$.]

Question 6. [20 points] Consider the following decision problems about grammars and the languages they generate.

1. $\texttt{DP1}$: Given a context-free grammar G and a right-linear grammar G', is $L(G) \subseteq L(G')$?
2. $\texttt{DP2}$: Given a context-free grammar G and a right-linear grammar G', is $L(G') \subseteq L(G)$?

For each decision problem determine whether it is decidable or undecidable. If a decision problem is decidable give a decision procedure which decides it. You should only use sub-routines discussed in class. A proof of correctness is not required. If a decision problem is undecidable, give a reduction.

Supplementary Questions

These questions are not for credit. Do not submit them to `CrowdMark`. Nothing even remotely close to these questions will appear on the final, but they are quite fun so consider trying them in case you finish this assignment early.

Question 7. [0 points] Write a program in your favourite programming language that prints its own text exactly.

Question 8. [0 points] The Halting Problem and all the problems equivalent to it are CE complete. This means that for any CE complete problem P there is a mapping reduction $P \leq_m \texttt{HP}$. Are there any problems that are undecidable and CE but not CE complete? In other words, is there a problem that is undecidable but less difficult than the halting problem?