Comp 330 - Lecture 10 - October 3rd

- Italian expression: Inghiottire il rospo
  → swallow the toad
  → to tolerate something

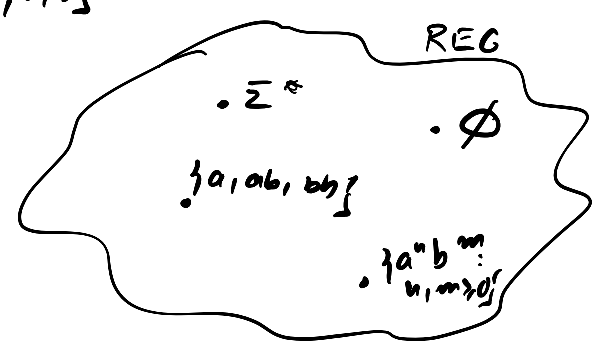
Cesare hopes that his lectures do not make his 330 students want to "inghiottire il rospo".

- Posted isomorphism proof for Lecture 9
- Discuss the midterm on Thursday - Yes this week's material will be on midterm.

Languages which are not regular → Tuesday. → Thursday.

$\Sigma = \{a, b\}$ $\qquad L \subseteq \Sigma^*$

So far
FAs,
RegEx,
Closure
Properties



$\{a^n b^n : n \geq 0\}$

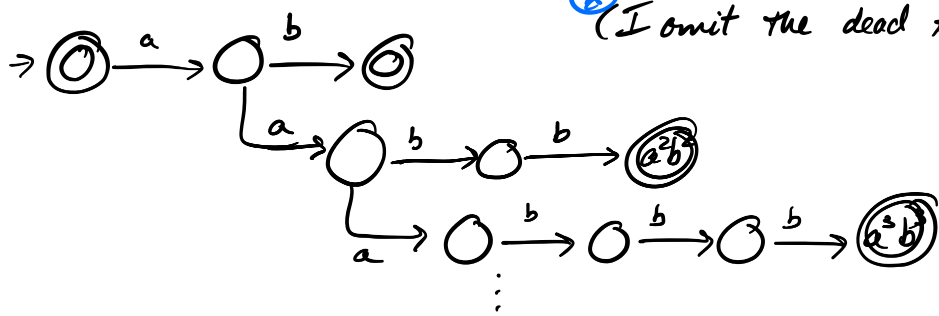
What does it mean if L is not regular?
→ DFA do not have enough "computational power" to accept L

→ But major limitation of DFA is that they must have a finite # of states

<u>For example</u> $L = \{ a^n b^n : n \geq 0 \}$

$\varepsilon \quad ab \quad a^2b^2 \quad a^3b^3 \quad \ldots$

(I omit the dead transitions)



DFA grows arbitrarily large, to accept L <u>need at least an infinite number of states</u>

Hmm, what theorem did we see recently that gives us the minimum # of states necessary to accept a regular language? The M-N Theorem.

In M-N, $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$

(1) L is regular <=> (2) Index of $\equiv_L$ is finite

OR

(1) L is not regular <=> (2) Index of $\equiv_L$ is infinite

But how do we show that index of $\equiv_L$

is infinite ?

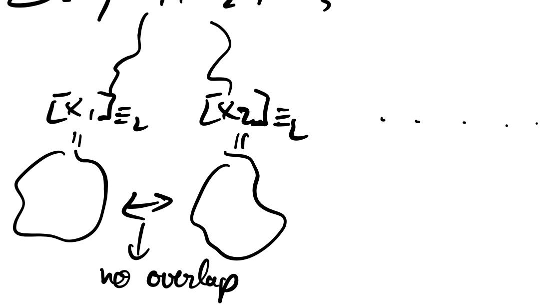
Proof Technique $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$. Use M-N to prove L not reg.

1. Construct an infinite set of strings $S \subseteq \Sigma^*$
   [S & L do not need to be related]

2. Show that $\forall x, y \in S$, $x \neq y$, $x \not\equiv_L y$
   i.e. find $z \in \Sigma^*$ s.t. $xz \in L$, $yz \notin L$
   [OR vice-versa]
   not necessarily in S, by def $\equiv_L$

3. Implies that every string in S must be in its own eq. class

$S = \{ x_1, x_2, x_3 \qquad \}$



$\therefore$ $\equiv_L$ has at least an infinite # of eq classes

$\Rightarrow$ index of $\equiv_L$ is infinite.

Ex Use M-N to prove that $L = \{a^n b^n : n \geq 0\}$ is not reg. ($\Sigma = \{a, b\}$)

1. $S = \{ a^i : i \geq 0 \}$

2. $x, y \in S$, $x \neq y \Rightarrow x = a^i$, $y = a^j$, $i \neq j$
   Find $z \in \Sigma^*$ s.t. $xz \in L$, $yz \notin L$
   $z = b^i$ (also b^j good)
   $xz = a^i b^i \in L$ but $yz = a^j b^i \notin L$
   same z

3. Each string in S is a rep. of a distinct eq class of $\equiv_L$ => $\equiv_L$ has inf. index.
=> L is not reg.

__Remark__ The eq classes $[\varepsilon]_{\equiv_L}, [a]_{\equiv_L}, [a^2]_{\equiv_L}, \ldots$ correspond to states any DFA had to have to accept L => Intuition checks out!

[9:15]

__Ex__ Use the M-N thm to show that
$L = \{ w \in \{a,b\}^* : w = w^R \}$ is not regular.

-> What is w^R? It's the reverse of w.

$w = abb \quad w^R = bba$

Recursive def: $\varepsilon^R = \varepsilon$, $\quad x \in \Sigma^*, a \in \Sigma$ $\quad (xa)^R = a x^R$

-> L is the set of all strings w s.t. $w = w^R$

$w = aba \quad w^R = aba$

-> Intuitively L should not be reg b/c DFA would need to remember first half of string & check that the second half is its reverse [Remember how to check for palindrome from 202? 250?]

Could be many things

1. $S = \{ a^i b : i \geq 0 \}$
2. $x = a^i b \quad y = a^j b$, $i \neq j$. WLOG, suppose $i < j$

let $z = a^i$

$xz \quad a^i b a^i \qquad yz = a^j b a^i$

$yz = \overbrace{a \ldots a}^{j}\, b\, \overbrace{a \ldots a}^{j}$ $\supseteq$ Reverse ✓ $\in L$

$a \ldots a\, b\, a \ldots a$

$xz = \overbrace{a \ldots a}^{i}\, b\, \overbrace{a \ldots a}^{j}$ $\supseteq$ Reverse

$\underbrace{a \ldots a}_{j}\, b\, \underbrace{a \ldots a}_{i}$ → clearly different $\notin L$

3. Index of $\equiv_L$ is infinite

[9:30]

M-N can only be used to show a language is not reg. But remember the point of this course is to prove & identify problems which are not decidable, a much more general category of languages. To do so, will need more tools: The Pumping lemma → ×REG, ×CFL & reductions ×Decidable.

The Pumping lemma (for regular languages)

Illustration (DO NOT DO THIS IN YOUR SOLUTION)

- Suppose L is an infinite language & $\exists$ DFA M with 3 states s.t. $L = L(M)$

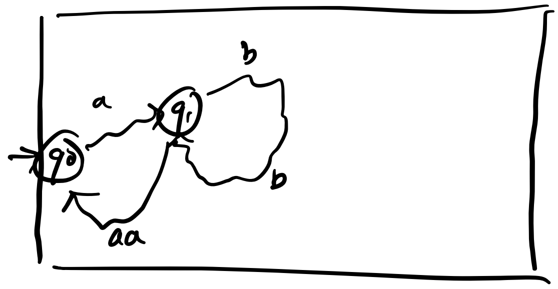
  [Why 3? It's just an example]

- Now suppose $w = abbaa \in L$, notice how $|w| \geq 3$, what happens in M?

$$q_0 \xrightarrow{a} q_1 \xrightarrow{b} q_2 \xrightarrow{b} q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_0$$

MUST be a state you've already visited ⇒ Using PHP { pigeons : letters, holes : states

In M Have a cycle $q_1 \xrightarrow{bb} q_1$



But now, in general, if start by reading a, can read bb as many times as I want, & $a(bb)^i aa \in L$

This is exactly what P.L. for reg languages tells us.

The PL for reg languages $\quad \Sigma \neq \emptyset, \ L \subseteq \Sigma^*$

if L is regular then

$\exists \ p \in \mathbb{N}, \ p > 0.$ [Think of this as the # of states in DFA accepting L]

$\forall \ w \in L, \ |w| \geq p.$ [We find a non-empty cycle in M w/in first p letters]

$\exists \ x, y, z \in \Sigma^*, \ w = xyz, \quad |xy| \leq p, \ |y| > 0$

$\forall \ \underbrace{i \in \mathbb{N}}_{\text{number times pass } y \text{ through cycle}}, \ w_i = xy^iz \in L$

→ Note, i can be 0.

Pf Already did most of it.

L is reg $\Rightarrow \exists$ DFA M s.t. $L(M) = L$. let $p = |Q|$

$w \in L, \ |w| \geq p \Rightarrow$ within first p letters w will have revisited all states

$$\underbrace{q_0 \xrightarrow{x} q_i \xrightarrow{y} q_i}_{|xy| \leq p} \xrightarrow{z} \circledcirc$$

$w = x \ y \ z$ Then can repeat y as many times

as I want (including none)
$$\begin{tikzcd} q_0 \arrow[r, squiggly, "x"] & q_i \arrow[loop below, "y"] \arrow[r, squiggly, "z"] & \circledcirc \end{tikzcd} \quad xy^iz \in L, \forall i \in \mathbb{N}$$

[This proof is slightly diff then from the .]
[lecture - What happens if L is finite]

How is this helpful ~~to show~~ that a language L is not regular?

$$L \text{ is reg} \Rightarrow PL \text{ is satisfied}$$
$$\neg PL \Rightarrow L \text{ is not reg.}$$

Let's negate the PL:

Recall $\neg \forall x \; P(x) \Rightarrow \exists x \; \neg P(x)$

$\neg \exists x \; P(x) \Rightarrow \forall x \; \neg P(x)$

$\forall \; p \in \mathbb{N}, \; p > 0.$

$\exists \; w \in L, \; |w| \geq p.$

$\forall \; x, y, z \in \Sigma^* \quad w = xyz, \; |xy| \leq p, \; |y| > 0$

$\exists \; i \in \mathbb{N}, \; w_i = xy^iz \notin L$

We will see a proof technique where we play the $\exists$ quantifier & we play against the $\forall$ quantifier. If we can show that we have a winning strategy then $\neg PL$ is true & $\therefore L$ is not reg (Winning strategy stems from Propositional/Predicate calculus).