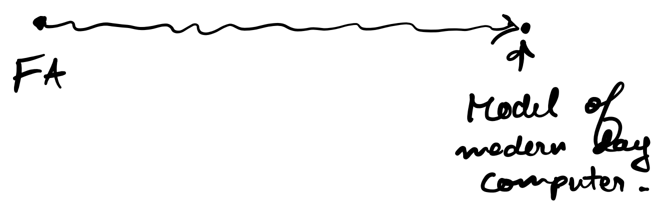
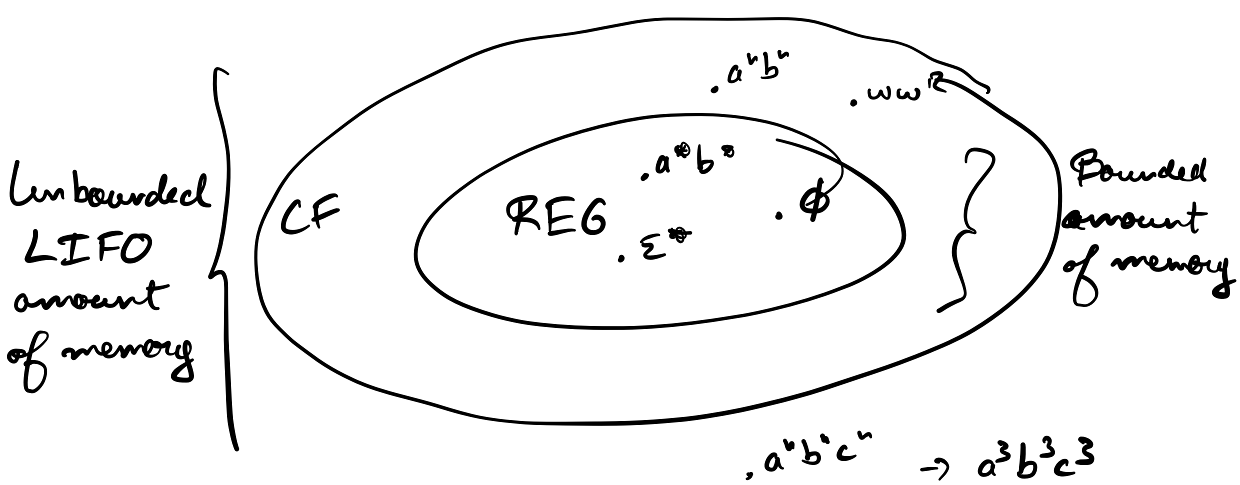
Comp 330 - Lec 17 - Oct 31st

IE: Acchiappare farfalle
To catch butterflies
To waste time

Languages which are not CF

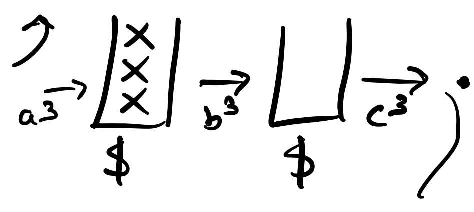
Automata theory:





$a^nb^nc^n$ $\rightarrow a^3b^3c^3$

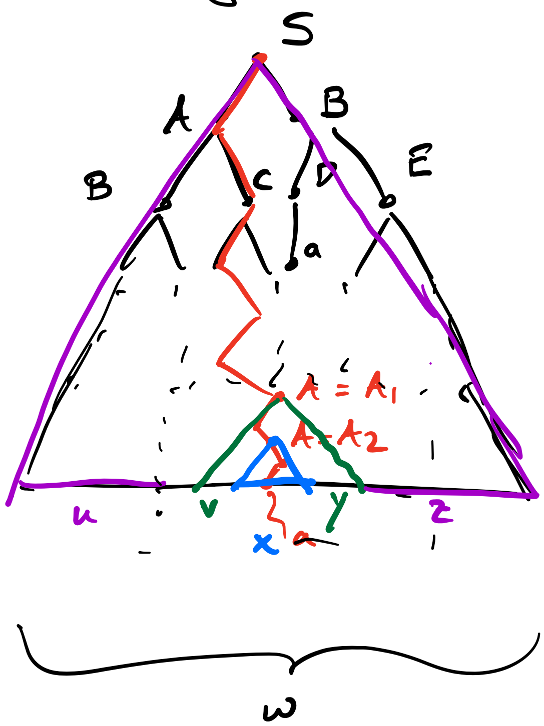
PDA which recognized

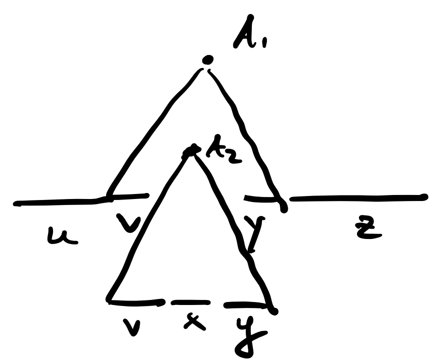


PDA has forgotten the # a's & b's

Pumping lemma for CFLs

If L is an infinite CFL then $\exists$ a CFG G in CNF s.t. $L(G) = L - \{\varepsilon\}$.
Since L is infinite & G is in CNF, for a sufficiently long string $w \in L$, the parse tree with yield w will look like:





$uv^2xy^2z \in L$

$\Downarrow$

$\forall i \; uv^ixy^iz \in L$.

Lemma (PT) Let $G = (V, S, T, P)$ be a CFG in CNF, $T_{G,w}$ be a parse tree of G with yield w and $n \in \mathbb{N}, n \geq 1$.

If the depth of $T_{G,w}$ is n then $|w| \leq 2^{n-1}$.

Pf By strong induction on n. Post later.

<u>Lemma</u> (PL for CFL) $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$

if L is CF then

$\exists\, p \in \mathbb{N},\ p > 0$

$\forall\, w \in L,\ |w| \geq p$

$\exists\, u, v, x, y, z \in \Sigma^*.$

$w = uvxyz,$

$|vy| \geq 1,$

$|vxy| \leq p$

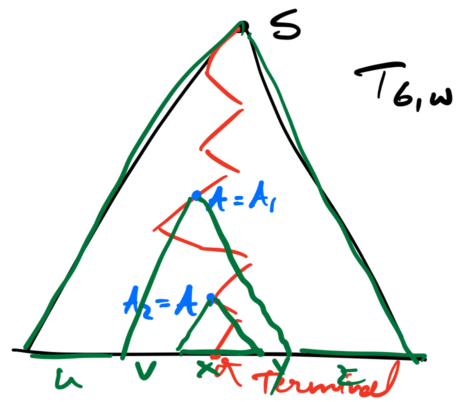
$\forall\, i \in \mathbb{N},\ w_i = uv^ixy^iz \in L.$

Pf

Setup $\exists, \forall$.

Suppose L is an infinite CFL.
Consider G as a CFG in CNF s.t.
$L(G) = L - \{\varepsilon\}$.
Let $k = |V|$ (# of variables of G)
Set $p = 2^k$, pick $w \in L$ $|w| \geq p = 2^k$.

Let $T_{G,w}$ be a parse tree of G with yield w. By the PT lemma, the depth of $T_{G,w}$ must be of length at least $k+1$.



Consider a longest path from the root S to a leaf of $T_{G,w}$.

The length of this path will be at least $k+1$.

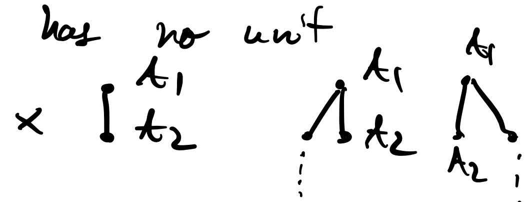
$\therefore$ It will contain at least $(k+1)+1$ nodes

$k+2$.

$\therefore$ At least $k+1$ variables will be in this path.

By PHP, there must a variable which is repeated at least once. Let A be the repeated variable, searching bottom-up, assign the variables A_1 & A_2. Partition $T_{G,w}$ based on the parse trees rooted at A_1 & A_2. This allows me to construct substrings u, v, x, y, z.

$|vy| \geq 1$ because G has no unit productions



$\exists$

$|vxy| \leq p$ because the repeated variable would have occurred by the $k+2^{nd}$ node. At most the parse tree rooted at A_1 would have depth $k+1$. By PT lemma,

$|vxy| \leq 2^{(k+1)-1} = p$

$\forall$ By copying Δ^{A_1} onto $\cdot A_2$ i times, create a new valid parse tree of G with yield $w_i = uv^ixy^iz \in L$.

Technically it's $i-1$ times

Using PL to show L is not CF

if L is CF then PL conditions are true

Contra-positive

if PL conditions do not hold then L is not CF.

Take contrapositive:

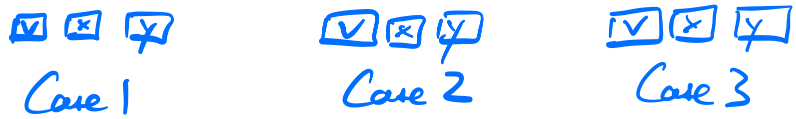
if $\forall p \in \mathbb{N},\ p > 0$
$\exists\ w \in L,\ |w| \geq p$
$\forall\ u, v, x, y, z \in \Sigma^*$,
$w = uvxyz,\quad |vxy| \leq p,\quad |vy| > 0$
$\exists\ i \in \mathbb{N}\quad w_i = uv^ixy^iz \notin L$
Then L is not CF.

Ex Prove $L = \{a^nb^nc^n : n \in \mathbb{N}\}$ is not CF.

$\forall$: Opponent picks $p \in \mathbb{N},\ p > 0$

$\exists$: $\quad w = a^p b^p c^p \in L$, $|w| = 3p \geq p$

$\forall$: $\quad w = \overbrace{a a \dots a a}^{p}\ \overbrace{b b \dots b b}^{p}\ \overbrace{c c \dots c c}^{p}$



Case 1 $\qquad v = a^{k_1} \quad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq p$

$|vxy| \leq p$

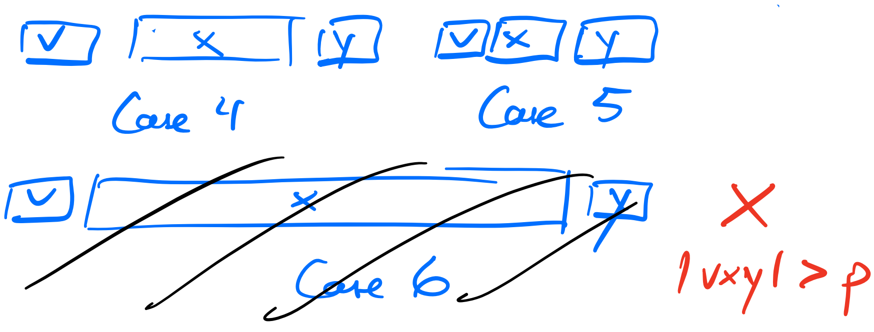
Pick i s.t. $\quad w_i = u \underbrace{v \dots v}_{i} x \underbrace{y \dots y}_{i} z \notin L$

$i = 2 \qquad w_2 = a^{p+k_1+k_2} b^p c^p \notin L$

$p + k_1 + k_2 \geq p + 1 > p$

Case 2/3 : Similar argument as Case 1

$w = \overbrace{a a \dots a a}^{p}\ \overbrace{b b \dots b b}^{p}\ \overbrace{c c \dots c c}^{p}$



Case 4 : $\qquad v = a^{k_1} \qquad y = b^{k_2} \qquad 2 \leq k_1 + k_2 \leq p$

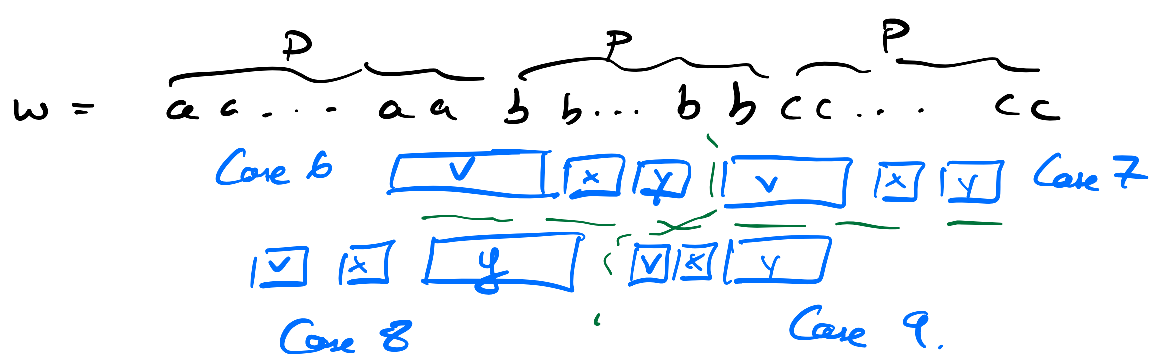
$k_1, k_2 \geq 1$

$\exists \quad i=2 \qquad w_2 = u v^2 x y^2 z$

$$= a^{p+k_1} b^{p+k_2} c^p \notin L$$

$$p+k_1 \geq p+1 > p.$$

Case 5 : Similar to Case 4.



Case 6 : $\exists \quad v = a^{k_1} b^{k_2} \quad y = b^{k_3}$

$$k_1, k_2 \geq 1$$

$$2 \leq k_1 + k_2 + k_3 \leq p$$

$i=2 \qquad w_2 = \dots \underbrace{a \dots a}_{k_1} \underbrace{b \dots b}_{k_2} \underbrace{a \dots a}_{k_1} \underbrace{b \dots b}_{k_2} \dots \notin L$

Pattern mismatch

Case 7/8/9 : Similar arg to Case 6. ∎

Ex Suppose $\Sigma \neq \emptyset$, $|\Sigma| \geq 2$.

Is $L = \{ww : w \in \Sigma^*\}$ CF ?

$ab\,ab$ $\quad abb\,aba \notin L$

No! $w = a^p b^p a^p b^p$

Q: What if $|\Sigma| = 1$?

Ex Is $L = \{a^{n^2} : n \in \mathbb{N}\}$ is CF?

No! $w = a^{p^2}$ → Argument would be exactly as the one for REG

Thm If $L \subseteq \{a\}^*$. If L is not regular then L is not CF.

Pf if L is CF then L is REG.
Exercise: Spiritual growth.

Closure properties of CFL – take 2

Ex $L_1 = \{a^n b^n c^m : n, m \in \mathbb{N}\}$

$L_2 = \{a^m b^n c^n : n, m \in \mathbb{N}\}$

L_1 & L_2 are CF. Please check!

$$L_1 \cap L_2 = a \ldots a\, b \ldots b\, c \ldots c$$

$$= \{a^n b^n c^n : n \in \mathbb{N}\}$$

$\Rightarrow$ CFLs are not closed under the $\cap$.

CFLs are closed under the REG $\cap$

Thm $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$, $R \subseteq \Sigma^*$,
If L is CF & R is REG then
$L \cap R$ is CF.

Pf DFA M s.t. $L(M) = R$
PDA P s.t. $L(P) = L$

Create a PDA $P' = (Q', \Sigma, \Gamma, \delta', s', Z, F')$

$Q' := Q_P \times Q_M$

$\Sigma := \Sigma$

$\Gamma := \Gamma_P$

$s' := (s_P, s_M)$

$Z := Z_P$

$F' := F_P \times F_M$

$$Q' \times \Sigma \times \Gamma \to Q' \times \Gamma^*$$

$$\delta'((p, q), \sigma, A) = ((\delta_P(p,\sigma,A)[1], \delta_M(q,\sigma)),$$

$Q_P \; Q_M$ $\qquad\qquad \delta_P((p, \sigma, A)[2] \qquad)$

Product construction.

Who cares?

Ex Show that CFL are NOT closed under the complement.

$L = \{a^n b^m c^k : n \neq m \text{ OR } m \neq k\}$

L is CF. Check why. 4 branches

$\overline{L} \cap L(a^* b^* c^*) = \underbrace{\{a^n b^n c^n : n \in \mathbb{N}\}}_{\Downarrow \atop \text{CF}}$

$a^n b^n c^n \quad bbaacca\text{-}a$

CFL are not closed under the complement.

Remark $\qquad L = \{ww : w \in \{a, b\}^*\} \Rightarrow$ NOT CF

$\overline{L}$ is CF $\Rightarrow$ Design a PDA

$\Rightarrow$ VALCOMPs.