David Hilbert 1900

10^{th} PROBLEM: Give a mechanical procedure to determine whether a given Diophantine equation has a solution.

$n > 2$ $x^n + y^n = z^n$

1994 Wiles

What is meant by a mechanical procedure?

KURT GÖDEL

ALONZO CHURCH 1935

λ-calculus

ALAN TURING 1936

Turing machine

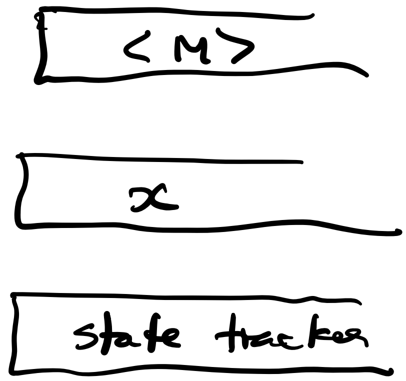
Church & Turing showed the existence of unsolvable problems.

Variations: 2 tapes, more tapes
2D tape, nondeterminism, probability,
quantum resources ?
None of these increase what
you can compute

Church-Turing thesis:
Any formalism for defining
computations is $\equiv$ TM.

Universal TM

A TM M has a
<u>description</u> which is just
a <u>finite</u> string.
$\langle M \rangle$: the <u>description</u> of
a TM.
$\langle M, x \rangle$: description of
M together with input x.



The Halting Problem is unsolvable.

There is no program that can read the text of another program P and input x for P and __decide__ whether P halts on input x.

__decide__ $\rightarrow$ YES
$\rightarrow$ NO

S program $\quad \langle S \rangle$: text of S

$S(x)\downarrow$ halts when S is run on x

$S(x)\uparrow$ loops forever

Suppose $\exists\ H(\langle S \rangle, x)$ s.t

$$H(\langle S \rangle, x) \begin{cases} \rightarrow \text{YES} & \text{if } S(x)\downarrow \\ \rightarrow \text{NO} & \text{if } S(x)\uparrow \end{cases}$$

H itself __always__ halts.

$\overline{H}(x) \equiv$ if $H(x, x)$ then loop else halt.

What happens to $\overline{H}(\langle \overline{H} \rangle)$

$\overline{H}(\langle \overline{H} \rangle) \rightarrow$

if $\underbrace{H(\langle \overline{H} \rangle, \langle \overline{H} \rangle)}$ then loop else halt

Does $\overline{H}(\langle \overline{H} \rangle)$ halt?

$\overline{H}(\overline{H}) \xrightarrow{\text{YES}}$ if H says YES it loops

$\overline{H}(\overline{H}) \xrightarrow{\text{NO}}$ if H says NO, it halts

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$L \subseteq \Sigma^*$ language

$M \rightarrow$ TM

$L(M) = \{ w \in \Sigma^* \mid M \text{ accepts } w \}$

Turing <u>recognizable</u> language.

L is Turing <u>decidable</u> if
$\exists$ TM M s.t $L = L(M)$ <u>and</u>
$\forall w \in \Sigma^*$, M halts on w.

L is computable enumerable if

$L = L(M)$ for some TM

L is computable if $L = L(M)$ for some TM M which always halts.

$$L_{HTM} = \{ \langle M, x \rangle \mid M \text{ halts on } x \}$$
$$\subseteq \Sigma^*$$

semi-decidable

$$f : \Sigma^* \longrightarrow \Sigma^*$$

Partial function

$$f : \Sigma^* \longrightarrow \Sigma^*$$

f maps a <u>subset</u> of Σ^* to Σ^*

f may be <u>undefined</u> on some inputs.

$\text{dom}(f) = \{ x \mid f(x) \text{ is defined} \}$

They compose like ordinary function.

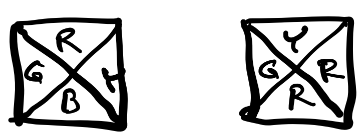
def A computable function is a function $f: \Sigma^* \longrightarrow \Sigma^*$ (not partial i.e. TOTAL) s.t $\exists$ a TM M s.t $\forall x \in \Sigma^*$

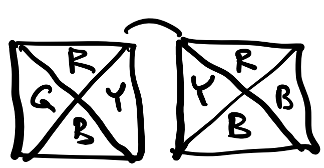
$$\vdash s\, x \,\square \cdots \xrightarrow{\;*\;} \vdash q\, f(x) \,\square \cdots$$

def A partial computable f^n is a partial f^n - : $\Sigma^* \longrightarrow \Sigma^*$ $\exists$ TM M s.t. $\forall x \in dom(f)$

$$\vdash s\, x \,\square \cdots \longrightarrow \vdash q\, f(x) \,\square \cdots$$

<u>and</u> $\forall x \notin dom(f)$ M will loop.
sometime called semicomputable.





UNDECIDABLE

WORD PROBLEM $g_1 g_2 \cdots g_n = g_1' \cdots g_m'$

TOPOLOGY Are two surfaces "equivalent"?

$$\left\{ \overset{M_1}{\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 6 \\ 2 & 1 & 3 \end{pmatrix}}, \dots\dots \overset{M_k}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}} \right\}$$

finite set of 3×3 matrices

Q: Is there some sequence (with repetitions) e.g. $M_{33} M_7^5 M_{33} M_2 M_5^{11}$ s.t the product is 0. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

UNDECIDABLE