Comp 330 - Lecture 22 - Nov 16th

IE : Prendere lucciole per lanterne

Extend A5 to Sunday at 11:59 PM

Characterization of undecidability (2)

HP : Undecidable but you can check yes-instances

T.R. / C.E.

EMPTY : Undecidable but you can check no-instances

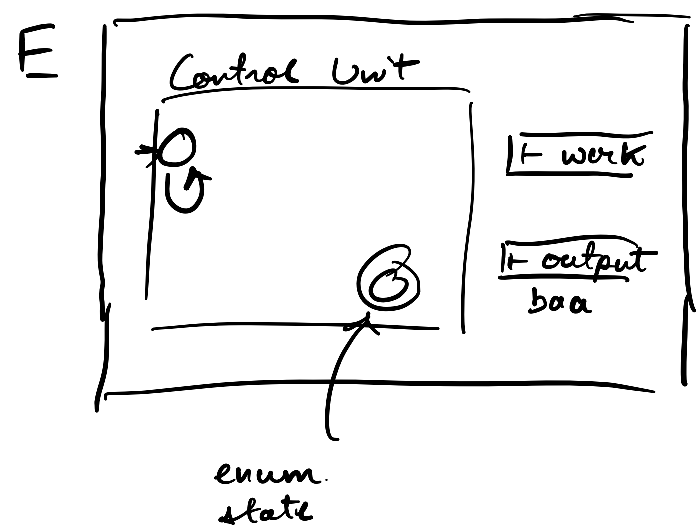
co-T.R. / co-C.E.

EQ : Undecidable but you can neither check a yes-instance nor a no-instance

neither C.E. nor co-C.E.

Enumerator

E runs forever



→ abb baa ....

$\uparrow$ $\uparrow$

w_1 w_2

$L(E) = \{ abb, baa, \ldots, w_i, \ldots \}$

E enumerates w_i in some finite # of steps

aba could not be enumerated
abb could be enumerated multiple times

$L(E) = \emptyset$ if E is in ∞-loop and never reaches the enum state

Def $\Sigma \neq \emptyset$, an enumerator E is a comp. model which enumerates strings in Σ^* 0 or more times. E runs forever.

$L(E) = \{ w \in \Sigma^* : w \text{ is enumerated by } E \text{ in a finite \# of steps} \}$

Def $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$, L is computably enumerable (CE) if $\exists$ enumerator E s.t. $L(E) = L$.

Thm $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$

L is CE $\Leftrightarrow$ L is Turing recognizable

($\Rightarrow$) L is CE $\therefore$ $\exists$ enum E s.t. $L(E) = L$

W.T.S. L is recognizable

Design a TM M s.t. $L(M) = L$

$M :=$ On input w

1. Run E
2. For each string enumerated by E, x, check if $w = x$
3. If yes, accept.

$L(M) = L$

($\Leftarrow$) L is recognizable $\therefore$ $\exists$ TM M s.t. $L(M) = L$

W.T.S. L is C.E.

Design an enumerator E s.t. $L(E) = L$.

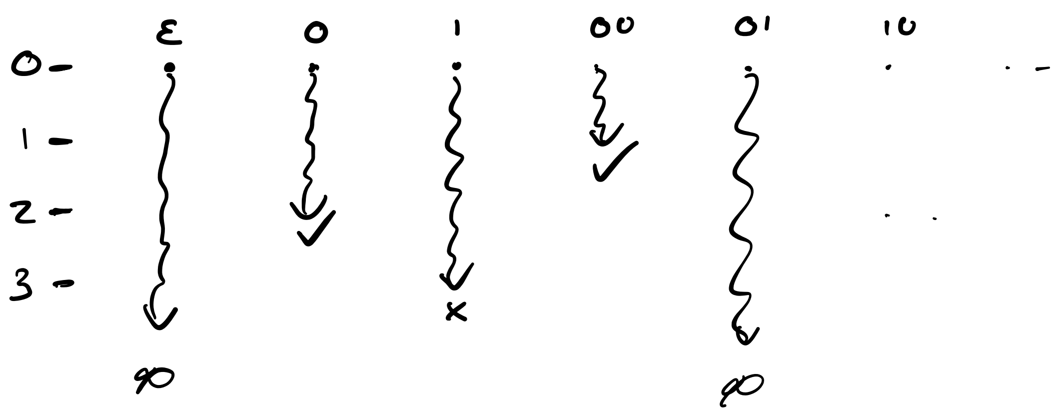
lexicographic order

$E :=$ 1. For $x \in \Sigma^*$

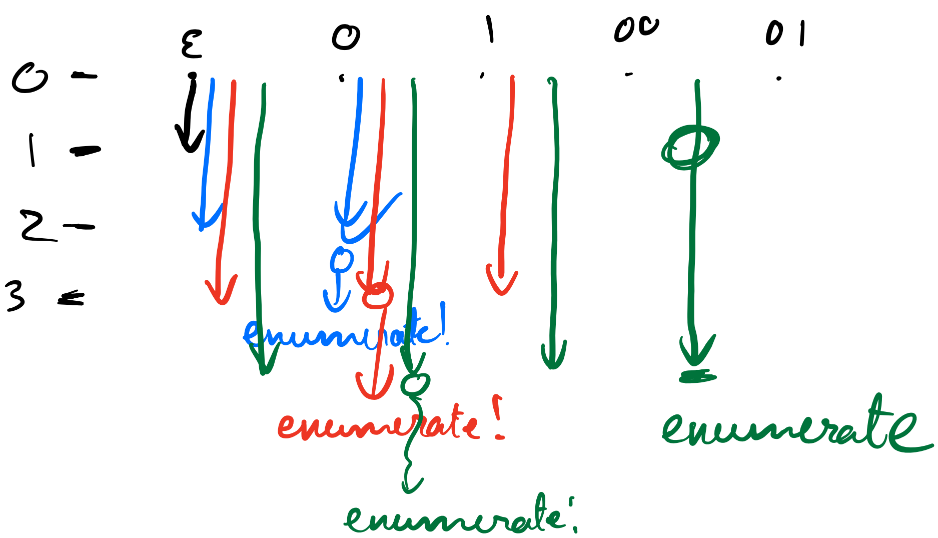
2. Run M on x

3. If M accepts x, then enumerate

M loops ∞ on ε
M accepts 0



Simulate ∞-parallelism with dovetailing / timesharing: Run M sequentially on each string for progressively more time steps.



$$\Sigma^* = \{ \varepsilon, 0, 1, 00, \dots \}$$
$\varepsilon \to w_1$, $0 \to w_2$, $1 \to w_3$, ...

$E :=$ 1. For $i = 1, 2, 3, \dots$
2. Run M on strings w_1 to w_i for i steps
3. Enumerate every string w_1 to w_i which M accepted.

Dovetailing

$$L(E) = L = L(M).$$

We can use dovetailing to create yes/no-verifier.

Def $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$, L is co-CE if $\overline{L}$ is CE.

Intuition: If the answer to D.P. is no then I can verify this.

Ex Show that EMPTY is co-CE.

$$L_{EMPTY} = \{ \langle M \rangle : M \text{ is a TM } \& \ L(M) = \emptyset \}$$

$\overline{L_{EMPTY}}$ is CE $\Rightarrow$ Create TM N s.t. $L(N) = \overline{L_{EMPTY}}$

$\overline{L_{EMPTY}}$ = { invalid instances of $\langle M \rangle$ } (Not hard, handcode rules of $\langle M \rangle$)
$\in$

$\cup$ { $\langle M \rangle$: M TM , $L(M) \neq \phi$ }

Dovetail computation of M on Σ^*

N := On input $\boxed{w}$ → Input to TM

1. Check if $w = \langle M \rangle$
2. If no, accept
3. Otherwise,
4. For $i = 1, 2, 3, \ldots$
   5. Run M on $\underbrace{w_1 \text{ to } w_i}_{\text{N creates these strings}}$ in i steps
   6. If M accepted any string in w_1 to w_i, then accept

$L(N) = \overline{L_{EMPTY}}$

If M loops forever on every string
$\Rightarrow L(M) = \phi$

This shows that EMPTY is co-CE.
Could EMPTY also be CE? No!
Because EMPTY is undecidable.

Thm (Post) $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$

L is decidable $\iff$ L is CE & co-CE.

Pf $\Rightarrow$ By definition

$\Leftarrow$ L is CE $\Rightarrow$ $\exists$ enum E_1 for L

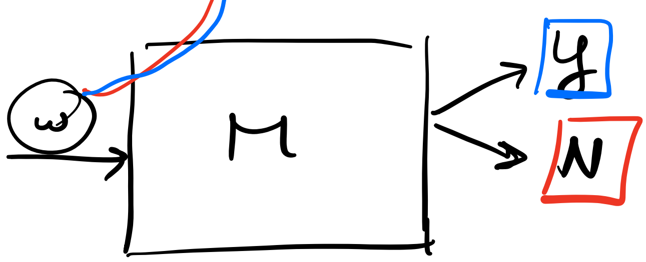
L is co-CE $\Rightarrow$ $\overline{L}$ is CE

$\Rightarrow$ $\exists$ enum E_2 for $\overline{L}$

$E_1 \to w_1 \quad w_2 \quad w_3 \quad \ldots \quad w \in L$

$E_2 \to u_1 \quad u_2 \quad u_3 \quad \ldots \quad w \in \overline{L}$

(*) E_1 & E_2 should be dovetailed



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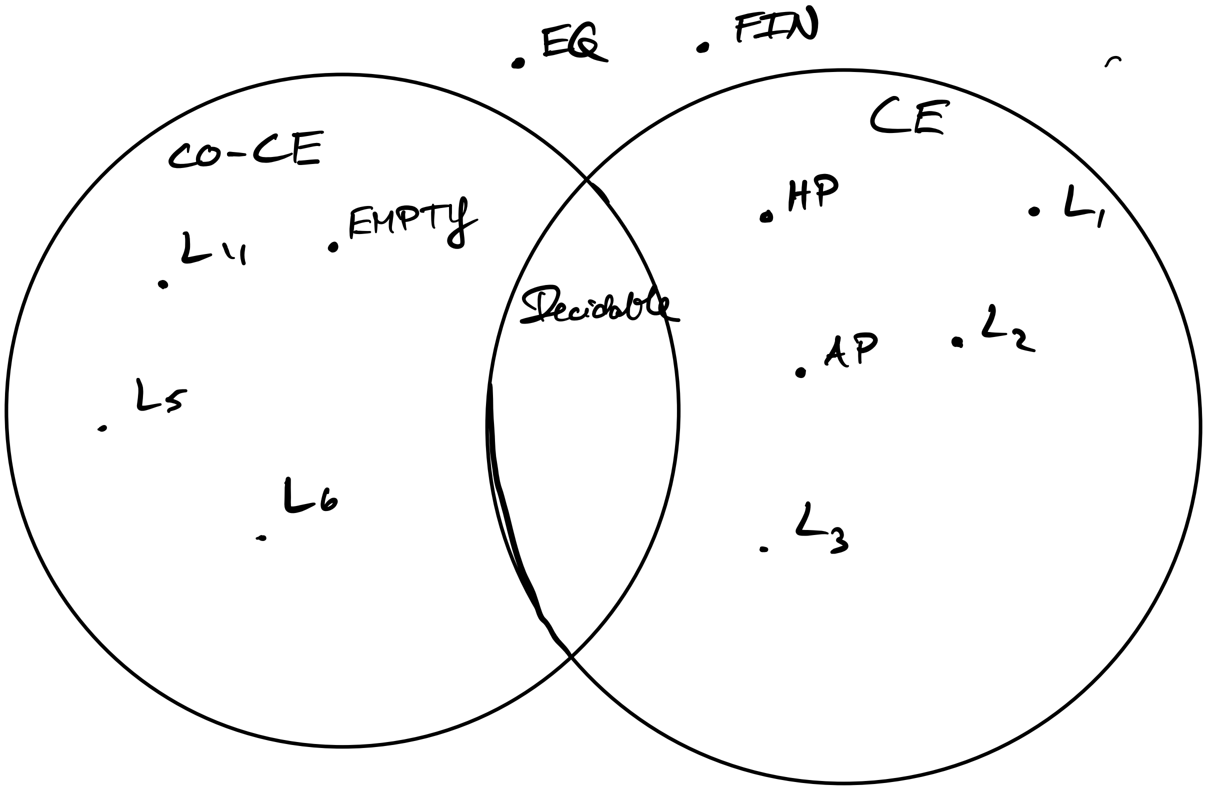
Implication Refine undecidability hierarchy



To show that L is neither CE nor co-CE we need 2 mapping reductions.

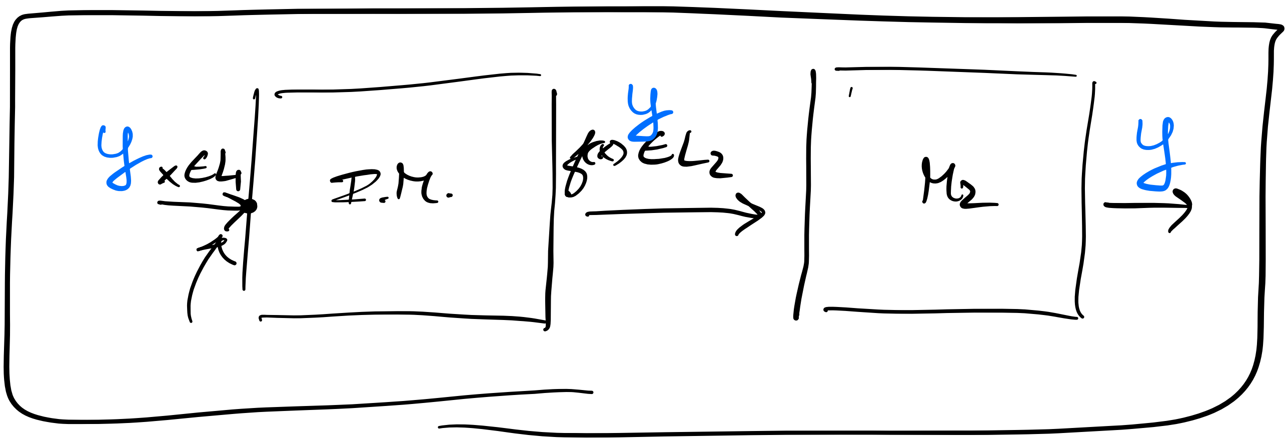
Thm $\Sigma, \Gamma \neq \emptyset$, $L_1 \subseteq \Sigma^*$, $L_2 \subseteq \Gamma^*$

$L_1 \leq_m L_2$, if L_2 is CE then L_1 is CE.

Pf $L_1 \leq_m L_2$ $\quad$ L_2 is CE

$\Rightarrow \exists$ TM M_2

$L(M_2) = L_2$



TM M_1 $L(M_1) = L_1$

$L_1 \leq_m L_2$

By contrapositive: if L_1 is not CE then L_2 is not CE

$\overline{L_1} \leq_m \overline{L_2}$ $\hookrightarrow$ if L_1 is not co-CE then L_2 is not co-CE

Ex $\quad$ HP $\not\leq_m$ EMPTY

HP $\downarrow$ not co-CE $\qquad$ EMPTY $\downarrow$ co-CE $\quad$ $\times$

EMPTY $\not\leq_m$ HP

EMPTY $\downarrow$ not CE $\qquad$ HP $\downarrow$ is CE $\quad$ $\times$

Ex Show that FIN is neither CE nor co-CE.

FIN $\langle M \rangle$: " Given a TM M,
is $\underbrace{|L(M)| < \infty}_{\text{is } L(M) \text{ finite.}}$?"

To show $\mathcal{R}$'s we need 2 mapping reductions:

A. $HP \leq_m FIN$ ← Less easy

HP: not co-CE; FIN: not co-CE

B. $\overline{HP} \leq_m FIN$ ← Easy

$\overline{HP}$: not CE; FIN: not CE

<u>Reduction A</u> $HP \leq_m FIN$

I_{HP} I_{FIN}

1. $\langle M, x \rangle \rightarrow \langle N \rangle$
2. M halts on x $\Leftrightarrow$ $L(N)$ is finite

Bounded computation: Use the time step at which M halted on x as the bound to the length of the strings accepted by N.

N := On input w
 1. Store w

2. Load x on tape
3. Run M on x for $|w|$ steps
4. If M has not halted on x by timestep $|w|$, then accept

   M is still running.
5. Else, reject.

$ANS(I_{HP}) = N \rightarrow M$ loops on x

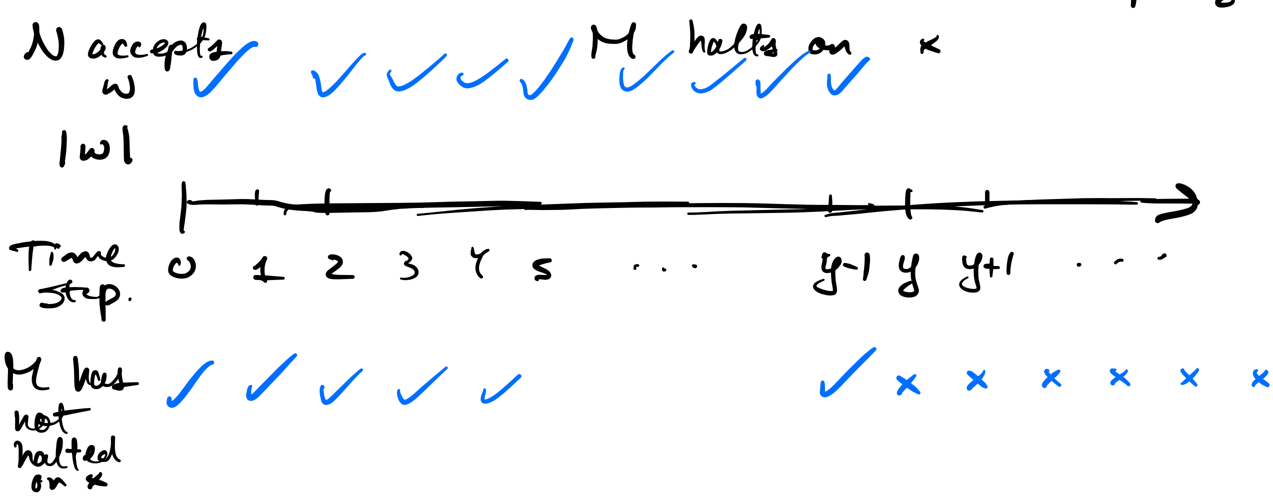
$\rightarrow$ Regardless of w, N accepts

$\rightarrow L(N) = \Sigma^*$

$\rightarrow ANS(I_{FIN}) = N$

$ANS(I_{HP}) = Y \rightarrow M$ halts on x

$\rightarrow$ At some time step y



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M is still running on x

$$L(N) = \{ u \in \Sigma^* : |u| \leq y-1 \}$$

$$|L(N)| < \infty$$

$$ANS(I_{FIN}) = Y$$

Reduction B

$$\overline{HP} \leq_m FIN$$

1. $\langle M, x \rangle \longrightarrow \langle N \rangle$
2. M loops on $x \rightarrow |L(N)| < \infty$

$N ::=$ On input w
1. Erase w & load x
2. Run M on x
3. If M halts on x, then accept

$ANS(I_{\overline{HP}}) = Y \rightarrow M$ loops on x
$\rightarrow$ Regardless of w, N loops
$\therefore$ N does not accept
$\rightarrow L(N) = \emptyset$
$\rightarrow |L(N)| = 0 < \infty$
$\rightarrow ANS(I_{FIN}) = Y$

$ANS(I_{\overline{HP}}) = N \rightarrow M$ halts on x
$\rightarrow L(N) = \Sigma^*$
$\rightarrow ANS(I_{FIN}) = N$