Comp 330 - Lecture 23 - Nov 21st

IE: Qualcosa bolle in pentola

Mercury course evaluation

A6 is out

Rice's Theorem Henry Gordon Rice - 1951

Ex T/F Given some D.P. A

A is CE $\iff$ $\bar{A}$ is co-CE

True $\bar{A}$ is co-CE $\iff$ $\bar{\bar{A}}$ is CE
$\iff$ A is CE

① HP is CE, $\underbrace{\text{is not co-CE}}_{\text{Post's Thm}}$

Via the fact above

② $\overline{HP}$ is co-CE, $\underbrace{\text{is not CE}}_{\text{Post's Thm}}$

Recall A, B

not CE not CE

$A \leq_m B$

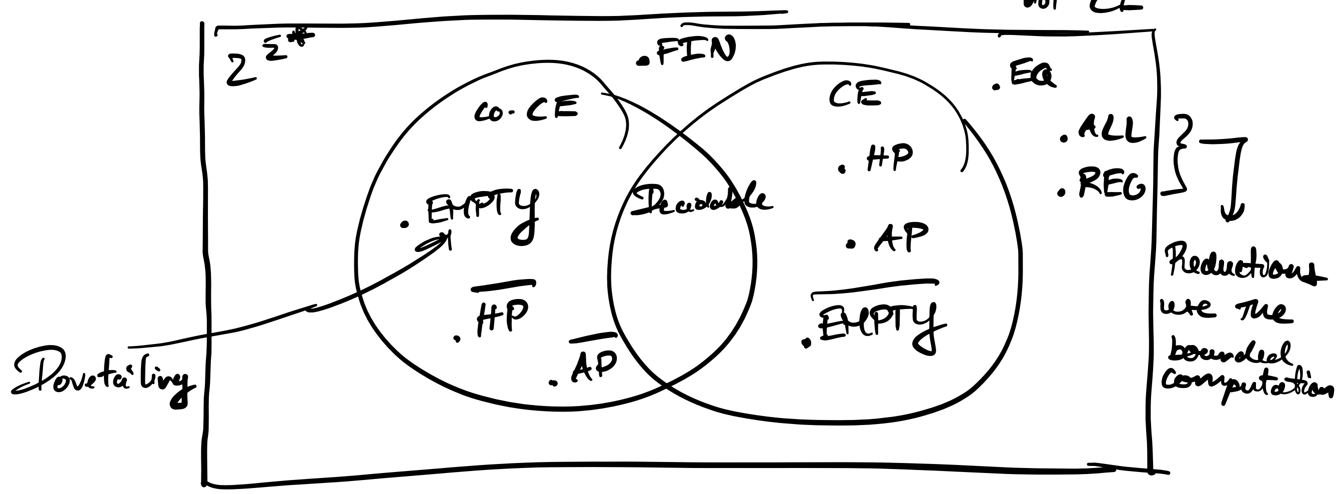
not co-CE not co-CE

not co-CE

At The end of Lec 22 $HP \leq_m FIN$ → Bounded computation

$\overline{HP} \leq_m FIN$

not CE



At this point we've seen many undecidable problems where the problem is about some property of $L(M)$

→ Question is about the language accepted by M, & not M itself

ALL → $L(M) = \Sigma^*$

REG → $L(M) = REG$

EMPTY → $L(M) = \phi$

FIN → $|L(M)| < \infty$

Not $L(M) \subseteq \Sigma^*$

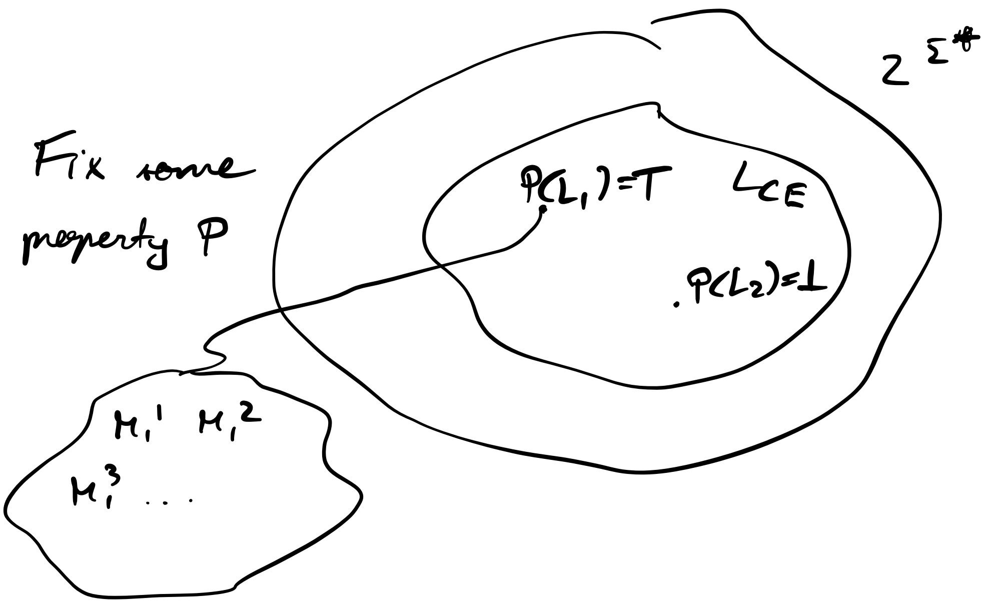
Are there any interesting decision problems about properties of CE languages which are decidable? No! Undecidability is the rule!

Impossibility is everywhere.

Def $\Sigma \neq \emptyset$, A property, P, of CE languages, L_{CE}, defined over Σ^* is a function

$$P : \overbrace{\{ L(M) : \exists \text{ TM } M \}}^{\text{CE languages}} \to \{ \overset{\text{True}}{\overset{\uparrow}{T}}, \overset{\text{False}}{\overset{\uparrow}{\perp}} \}$$

Fix some property P



We can re-write many of the DPs we showed were undecidable by designing a corresponding <u>property of CE languages.</u>

Ex $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$, L is CE

$$P_{EMPTY}(L) = \begin{cases} T & \text{if } L = \emptyset \\ \perp & \text{if } L \neq \emptyset \end{cases}$$

$EMPTY'$: " Given some TM M, is
$P_{EMPTY}(\cancel{M}) = T$? "
$L(M)$

$L_{EMPTY'} = \{ \langle M \rangle : P_{EMPTY}(L(M)) = T \}$

How do you check if a property is a property of CE languages? It should be implementation independent.

M is a TM s.t. $L(M)=L$
a language L where property P_M of

Thm $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$,

↳ Alternative def to property of a CE language

P_M is a property of CE languages $\iff$ $\forall$ TM M_1, M_2 if $L(M_1) = L(M_2)$ then $P_{M_1}(L(M_1)) = T \iff P_{M_2}(L(M_2)) = T$

The TM which accepts that language

language

$P(L(M)) \leftrightarrow P_M(L(M))$

Pf Otherwise, P would not be well-defined since it would return $T/\bot$ depending on $L(M_1)$ or $L(M_2)$

Ex Given some TM M

$$P_{330}(L(M)) = \begin{cases} T & \text{if } M \text{ accepts } 0^{330} \\ \bot & \text{otherwise} \end{cases}$$

P_{330} is a property of CE languages.

M_1, M_2 s.t. $L(M_1) = L(M_2)$

W.T.S. $P(L(M_1)) = T \iff P(L(M_2)) = T$

$$
\begin{aligned}
& P(L(M_1)) = T \\
\iff & M_1 \text{ accepts } 0^{330} \\
\iff & 0^{330} \in L(M_1) \\
\iff & 0^{330} \in L(M_2) \\
\iff & M_2 \text{ accepts } 0^{330} \\
\iff & P(L(M_2)) = T
\end{aligned}
$$

<u>Ex</u> Given a TM M

$$
P'_{330}(L(M)) = \begin{cases} T & \text{if } M \text{ rejects } 0^{330} \\ \bot & \text{otherwise} \end{cases}
$$

P'_{330} is not a property of CE languages

$M_1 :=$ On input w

1. If $w = 0^{330}$, rejects
2. otherwise, accepts

$L(M_1) = \Sigma^* - \{0^{330}\}$

$M_2 :=$ On input w

1. If $w = 0^{330}$, loop
2. otherwise, accepts

$L(M_2) = \Sigma^* - \{0^{330}\}$

$$\underbrace{\underbrace{P(L(M_1)) = \top} \quad \underbrace{P(L(M_2)) = \bot}}_{L(M_1) = L(M_2)}$$

__Exercise__ Given TM M, is P a property of CE languages?

$$P(L(M)) = \begin{cases} \top & \text{if } M \text{ halts on every } \underbrace{x}_{\text{input string}} \\ \bot & \text{otherwise.} \end{cases}$$

See my extra video which clarifies notation.

$$\boxed{\cancel{\text{if } \exists \text{ TM } M \text{ s.t. } L(M) = L \text{ \& } M \text{ rejects } 0^{330}}}$$

We said every interesting property of CE languages was undecidable → We meant non-trivial properties of CE languages.

Def $\Sigma \neq \emptyset$, A property P of CE languages is trivial if

1. $\forall\ L \subseteq \Sigma^*$, L is CE $\Rightarrow P(L) = \top$
↳ Trivially true

OR

2. $\forall\ L \subseteq \Sigma^*$, L is CE $\Rightarrow P(L) = \bot$
↳ trivially false

Ex Given a TM M, is $L(M)$ CE?
↳ Trivially true because $L(M)$ is CE by definition

Thm (Rice's Thm) $\Sigma \neq \emptyset$, Every non-trivial property of CE languages is undecidable

i.e. $\forall$ non-trivial property of CE languages P
$L_P = \{ \langle M \rangle : P(L(M)) = \top \} \subseteq \Sigma^*$
is undecidable

Pf Let P be some non-trivial property of CE languages.
Assume that $P(\emptyset) = \bot$ WLOG because if $P(\emptyset) = \top$ then the following arguments would be symmetric. (Exercise: Check!)

Since P is non-trivial $\exists\ L \subseteq \Sigma^*$ s.t. L is CE & $P(L) = \top$.

Since L is CE, $\exists$ a TM K which accepts L i.e. $L(K) = L$. $P(L(K)) = T$

We will show that

$$L_{HP} \leq_m L_P$$

$L_P = \{ \langle N \rangle : N$ is a TM & $P(L(N)) = T \}$

1. Convert $I_{HP} = \langle M, x \rangle$ to
   $I_{LP} = \langle N \rangle$
2. Show that M halts on x $\Leftrightarrow$
   $P(L(N)) = T$.

M, x inputs $\quad$ K constant

$N :=$ On input w $\quad$ Assume that N has 2 tapes

1. Save w to tape 1
2. Load x onto tape 2 $\quad$ $(:) \Leftrightarrow .$
3. Run M on x on tape 2
4. If M halts on x, then run K on w on tape 1.
5. If K accepts w then accept

$\underbrace{\boxed{M \text{ halts on } x}}_{I_{HP}} \Rightarrow K$ runs on every w

$\Rightarrow K$ accepts $w \Leftrightarrow$

N accepts w

$\Rightarrow L(K) = L(N) = L$

$\Rightarrow P(L) = \underbrace{\boxed{T = P(L(N))}}_{I_{LP}}$

$\underbrace{\boxed{M \text{ loops on } x}}_{I_{HP} = \text{No}}$ $\Rightarrow$ N loops on every w

$\Rightarrow L(N) = \phi$ — property of CE languages

$\Rightarrow P(L(N)) = P(\phi)$

$= \boxed{\perp}$ $\Rightarrow$ No $= I_{LP}$

$\therefore L_{HP} \leq_m L_P$.

Who cares?

1. Can use Rice's Thm to show undecidability. How?

   W.T.S. A is undecidable

   To use Rice's Thm, create some $\underbrace{\text{non-trivial}}_{①}$ $\underbrace{\text{property P of CE languages}}_{②}$

   s.t. $L_A = \{ \langle M \rangle : P(L(M)) = T \}$

Ex Show using Rice's Thm that EMPTY is undecidable

Given some TM M

$$P_{EMPTY}(L(M)) = \begin{cases} \top & \text{if } L(M) = \emptyset \\ \bot & \text{otherwise} \end{cases}$$

① Non-trivial

$M_1 :=$ On input w $\quad L(M_1) = \emptyset \quad P(L(M_1)) = \top$

1. Rejects

$M_2 :=$ On input w $\quad L(M_2) = \Sigma^* \quad P(L(M_2)) = \bot$

2. Accepts

② Property of CE Languages

M_1, M_2 TM, $L(M_1) = L(M_2)$

W.T.S. $P(L(M_1)) = \top \iff P(L(M_2)) = \top$

$P(L(M_1)) = \top \iff L(M_1) = \emptyset \iff L(M_2) = \emptyset$
$\iff P(L(M_2)) = \top$

By Rice's Thm EMPTY is undecidable □

<u>Thursday</u> : • D.P. about CFGs

$L(G) = \Sigma^*$ is undecidable

Next → • Tuesday : PCP, last 10 mins to talk about final