Comp 330 - Lec 25 - Nov 28th

IE: Non vedo l'ora

I can't wait for x

- Emil Post 1946
- Toy system -1920s
- Turing reduction

Post correspondence problem (PCP)

Ex 1 $\Sigma = \{a, b\}$

$$S = \left\{ \underset{\textcircled{1}}{\begin{bmatrix} aba \\ bab \end{bmatrix}}, \underset{\textcircled{2}}{\begin{bmatrix} aab \\ aa \end{bmatrix}}, \underset{\textcircled{3}}{\begin{bmatrix} a \\ aba \end{bmatrix}} \right\}$$

PCP: Given S, can we arrange the tiles ①, ② & ③ s.t. the strings of the top tiles concatenated = the strings of the bot tiles concatenated together.

$$\begin{array}{ccccc} \textcircled{2} & \textcircled{1} & \textcircled{2} & \textcircled{3} & \\ aab & aba & aab & a & \text{Match!} \\ aa & bab & aa & aba & \text{Answer: Yes.} \end{array}$$

Ex 2

$$S = \left\{ \overset{\textcircled{1}}{\begin{bmatrix} abb \\ bab \end{bmatrix}}, \overset{\textcircled{2}}{\begin{bmatrix} aab \\ aa \end{bmatrix}}, \overset{\textcircled{3}}{\begin{bmatrix} a \\ aba \end{bmatrix}} \right\}$$

If you start with tile ① ×

a b b
b a b

If you start with tile ③ ×

a

a b a

If you start with tile ②

a a b | a b b | a b b |
a a | b a b | b a b |
② ① ①

Need to stick tile ① forever

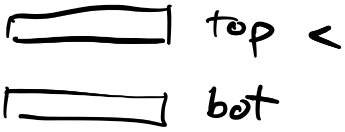
Answer is no!

Ex 3

$$S = \left\{ \begin{bmatrix} a\ a \\ a\ a\ b \end{bmatrix}, \begin{bmatrix} a \\ a\ b \end{bmatrix}, \begin{bmatrix} b\ a \\ b\ a\ b \end{bmatrix} \right\}$$

Is this instance solvable? No!

Because the length of top < bot



We can develop some heuristics, but that's the best you can do! PCP is undecidable.

Def (PCP) Given a finite set S, $|S| \geq 1$, $\Sigma \neq \emptyset$, $S = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$ $\forall i$ $\alpha_i, \beta_i \in \Sigma^+$, is there a sequence of integers $i_1 \ldots i_m \in [1 \ldots |S|]$, $m \geq 1$, s.t. $\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_m} = \beta_{i_1} \cdot \ldots \cdot \beta_{i_m}$.

An instance of PCP is a PCP-Set S.

$\langle S \rangle$ string representation of S.

$ANS(\langle S \rangle)$ = Yes if can find $i_1 \ldots i_m$ &

= No otherwise

Connection b/w PCP & computation ?

Given some comp. model, create tiles

start tile ⇕ starting config

transition tiles ⇕ δ

copy tiles / catch-up tiles ⇕ see example :)

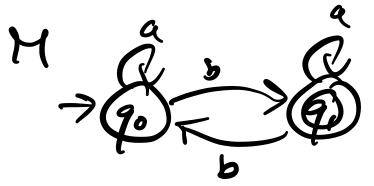
Creating a solution to MPCP ≡ Simulating some acceptance computation

Modified PCP (MPCP) = In addition to S, we specify which tile should always start a solution

① MPCP is undecidable => AP $\leq_m$ MPCP

② PCP is undecidable => MPCP $\leq_m$ PCP

<u>Ex</u> DFA M $x = ab$



$q_0\, a b \to \cancel{a}\, q_0\, b \to q_1$

$\# \underbrace{q_0\, a b}_{\alpha_0} \# \underbrace{q_0\, b}_{\alpha_1} \# \underbrace{\boxed{q_1}}_{\alpha_2} \#$ accept state

MPCP Solution $\approx \begin{array}{l} \# \; q_0 \;\; a \;\; b \;\; \# \;\; q_0 \;\; b \;\; \# \;\; q_1 \;\; \# \;\; \# \\ \# \; q_0 \;\; a \;\; b \;\; \# \;\; q_0 \;\; b \;\; \# \;\; \underbrace{q_1}_{} \;\; \# \;\; \# \end{array}$

catch-up tile

$\alpha_2 \to$ accept config

Starting tile

$$\begin{bmatrix} \# \\ \# \; \underbrace{q_0 \;\; a \;\; b}_{\alpha_0} \; \# \end{bmatrix}$$

Transition tile

$$\begin{bmatrix} q_0 \; a \\ q_0 \end{bmatrix} \quad \begin{bmatrix} q_0 \; b \\ q_1 \end{bmatrix}$$

Copy tiles

$$\begin{bmatrix} b \\ b \end{bmatrix} \begin{bmatrix} \# \\ \# \end{bmatrix}$$

Catch-up tiles

$$\begin{bmatrix} q_1 \# \# \\ \# \end{bmatrix}$$

We can do the same for TMs:

$$\langle M, x \rangle \to \langle S, (\theta, \beta) \rangle$$

s.t. M accepts x $\iff$ $\exists$ a solution to $S, \begin{pmatrix} \theta \\ \beta \end{pmatrix}$

solution $\equiv \# \alpha_0 \# \alpha_1 \# \alpha_2 \# \cdots \# \underset{\uparrow}{\alpha_N} \#$

accept config,

Thm PCP is undecidable but CE

Pf $\quad AP \le_m MPCP \le_m PCP$

↳ Post

TM M & input x

Transition + copy

Catch-up tiles

$$\text{MPCP Solution} \simeq \begin{array}{l} \# \boxed{\alpha_0} \# \alpha_1 \# \cdots \quad \# \alpha_{N-1} \# \\ \# \alpha_0 \# \boxed{\alpha_1} \# \quad \cdots \# \alpha_{N-1} \# \alpha_N \# \end{array}$$

accept config

TM M $\qquad x = aba$

$$\delta(s,a) = (q_1, b, R)$$
$$\delta(q_1, b) = (q_2, a, L)$$
$$\delta(q_2, \underline{b}) = (q_a, b, R)$$

$$\# \vdash s\, a\, b\, a\, \square \# \vdash b\, q_1\, b\, a\, \square \# \vdash$$
$$\# \vdash s\, a\, b\, a\, \square \# \vdash b\, q_1\, b\, a\, \square \# \vdash q_2\, b\, a\, a\, \square \# \vdash$$

Start tile
$$\left[\begin{array}{c} \# \\ \# \vdash s\, x\, \square \# \end{array} \right]$$

Copy tiles
$$\left[\begin{array}{c} \gamma \\ \gamma \end{array} \right] \gamma \in \Gamma \qquad \left[\begin{array}{c} \# \\ \# \end{array} \right]$$

Transition tile
$$\left[\begin{array}{cc} q & a \\ b & p \end{array} \right] \iff \delta(q,a) = (p,b,R).$$

$c \in \gamma \quad \left[\dfrac{c\ q\ a}{p\ c\ b}\right]$

$\delta(q,a) = (p,b,L)$

$p \neq q_r.$

$\# \vdash b\, q_1\, b\ a\ \square\ \# \vdash q_2\, b\, a\, a\, \square\ \#$

$\# \vdash b\, q_1\, b\ a\ \square\ \# \vdash q_2\, b\, a\, a\, \square\ \# \underbrace{\vdash b\, q_a\, a\, a\, \square}_{\text{accept state in } \cancel{\alpha_2}\ \Rightarrow\ \text{accept config}}\ \#$

α_3 (typo)

Catch-up. ⇒ Post catch-up.
(See after)

$AP \leq_m MPCP \implies MPCP$ is undecidable

$\triangledown$

Final exam : Dec 12th at 9AM –
12 PM

Fieldhouse

Content

Q1 : Reg languages

Q2 : Classify $\leq$ REG
NOT REG, CF
NOT CF

Q3 : Classify
- DEC
- UNDEC but CE
- UNDEC but co-CE
- Neither CE nor co-CE

Q4 : 1. DEC vs 2. UNDEC

Q5 : T/F + JUSTIFICATION → 1/2 lines

Q6 - Free. Lec 13-26

- Crib sheet: double-sided, 12 point, handwritten.

- Review session: Dec 8th 1PM to 4:30PM
  ENGMC 204

  Extra OHs : Dec 11th 2 to 5PM

- Post final preparation guide

$a \square \# \vdash b q_1 b a \square \# \vdash q_2 b a a \square \#$

$a \square \# \vdash b q_1 b a \square \# \vdash q_2 b a a \square \underbrace{\# \vdash b q_a a a \square \#}_{\text{Play catch-up}}$

Idea: Make top catch-up with bottom by making tiles where append more on top vs on bot. Do this around a "pivot" → The q_a.

$\# \vdash b q_a a a \square \# \vdash q_a a a \square \# q_a a a \square \# q_a a \square \# q_a \square \#$

$\# \vdash b q_a a a \square \# \vdash q_a a a \square \# q_a a a \square \# q_a a \square \# q_a \square \# q_a \#$

↓ Copy tiles

↓ catch-up tiles

$$\begin{bmatrix}\gamma q_a \\ q_a\end{bmatrix} \ \forall \gamma \in \Gamma \quad + \begin{bmatrix} q_a \# \# \\ \# \end{bmatrix}$$

$$+ \begin{bmatrix} q_a \gamma \\ q_a \end{bmatrix} \ \forall \gamma \in \Gamma$$

↑ This is how the same situation as with the DFA.

$\# \, q_a \, \# \, \#$ Done!

$\# \, q_a \, \# \, \#$

MPCP $\leq_m$ PCP : Intuition is search space for MPCP is "smaller" than PCP.

Not so easy to do this!

$$I_{MPCP} = \langle S, \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \rangle \rightarrow I_{PCP} = \langle S' \rangle$$

s.t.

ANS (I_{MPCP}) = Yes $\Leftrightarrow$ ANS (I_{PCP}) = Yes

Suppose $*, \diamond \notin \Sigma$ (for S)

Define <u>string operations</u> :

$u = \sigma_1 \sigma_2 \ldots \sigma_n \ , \ \sigma_i \in \Sigma$

$*u := *\sigma_1 * \sigma_2 * \sigma_3 * \dots * \sigma_n$

$u* := \sigma_1 * \sigma_2 * \dots \sigma_n *$

$*u* := *\sigma_1 * \dots * \sigma_n *$

Create S' by

1. For each non-starting tile in S $\begin{bmatrix} \theta_i \\ \beta_i \end{bmatrix}$, add $\begin{bmatrix} *\theta_i \\ \beta_i * \end{bmatrix}$ to S'

2. For the starting tile $\begin{bmatrix} \theta \\ \beta \end{bmatrix}$ add $\begin{bmatrix} *\theta \\ *\beta* \end{bmatrix}$ to S'

3. This creates an imbalance, catch-up with $\begin{bmatrix} *\diamond \\ \diamond \end{bmatrix}$, add to S'

$Ans(\langle S, \tfrac{\theta}{\beta} \rangle) = Yes \iff Ans(\langle S' \rangle)$

Why does this work?

Suppose MPCP had a solution:

$$\underset{ba}{\theta}\ \underset{aa}{\theta_3}\ \underset{bba}{\theta_7}\ \underset{a}{\theta_4} = \underset{b}{\beta}\ \underset{aa}{\beta_3}\ \underset{ab}{\beta_7}\ \underset{baa}{\beta_4}$$

$$\begin{bmatrix} ba \\ b \end{bmatrix} \begin{bmatrix} aa \\ aa \end{bmatrix} \begin{bmatrix} bba \\ ab \end{bmatrix} \begin{bmatrix} a \\ baa \end{bmatrix}$$

Modify tiles s.t. this is same tile arrangement for PCP solution

$$\begin{bmatrix} * b * a \\ * b * \end{bmatrix} \quad \begin{bmatrix} * a * a \\ a * a * \end{bmatrix} \quad \begin{bmatrix} * b * b * a \\ a * b * \end{bmatrix}$$

① ② ③

$$\begin{bmatrix} * a \\ b * a * a * \end{bmatrix}$$ ④

can't start with any of these tiles b/c $* \neq a$, $* \neq b$

(by design)

Need catch-up tile ↓

① ② ③ ④

$$* b * a * a * a * b * b * a * a * \diamond$$

$$* b * \boxed{a} * a * a * b * b * a * a * \diamond$$

↑ bottom so should be able to match with same tile from MPCP solution