Limitation of mapping reductions

Recall Mapping / many-to-one reductions. P, Q DPs. $P \leq_m Q$.

→ If you can solve Q, then you can solve P.

Two steps:

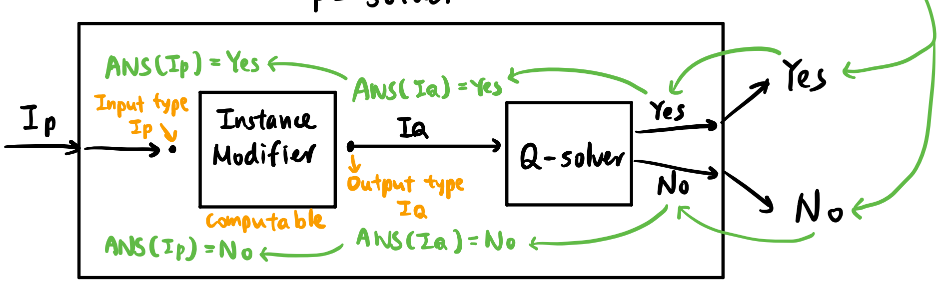
(1) Convert I_P (arbitrary) to I_Q (particular instance as a function of I_P.) in a computable manner then feed I_Q to the Q-solver to create a P-solver.

The answer of the Q-solver is the answer of the P-solver.

(2) Proof of correctness for the P-solver:

$ANS(I_P) = Yes \Leftrightarrow ANS(I_Q) = Yes.$

P- solver



Theorem $P \leq_m Q$.

· Q is decidable ⇒ P is decidable	P is undecidable ⇒ Q is undecidable.
· Q is CE ⇒ P is CE.	P is not CE ⇒ Q is not CE.
· Q is co-CE ⇒ P is co-CE.	P is not co-CE ⇒ Q is not co-CE.

Example Is there a mapping reduction s.t. $\overline{HP} \leq_m HP$?

Sol No. $\overline{HP}$ (co-CE, not CE) $\cancel{\leq_m}$ HP (CE, not co-CE.). But then the intuition of "at least as difficult" breaks down. Solution: Turing Reduction.

Turing Reductions

$P, Q, P \leq_T Q$.

(1) Convert I_P to I_Q and use Q-solver $\equiv$ Oracle for Q in any computable ways (e.g., use the Q-solver 10 times, flip answers, ...) to create a P-solver.

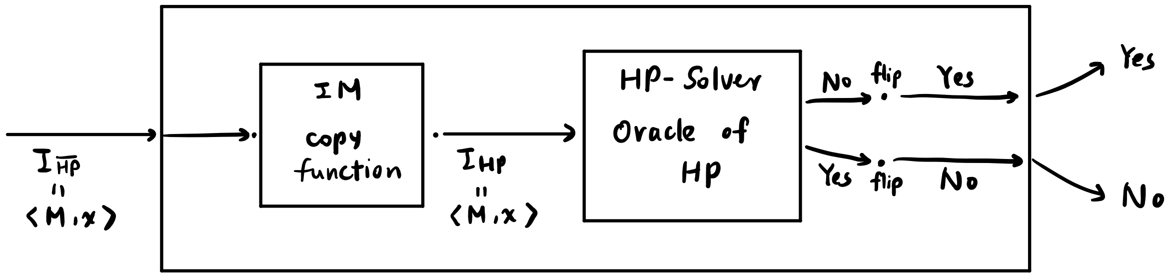
(2) $ANS(I_P) = $ Yes $\Rightarrow$ P-solver returns Yes.

$ANS(I_Q) = $ No $\Rightarrow$ P-solver returns No.

This is I_P

Example Show $\overline{HP} \le_T HP$.

Sol $\overline{HP}$ solver = Oracle TM with Oracle for HP



Proof of correctness for $\overline{HP}$-solver:

$ANS(I_{\overline{HP}}) = $ Yes $\Rightarrow$ M loops on x $\Rightarrow$ HP solver says No

Does M not halt on x?
Does M loop on x?

$\Rightarrow$ Answer gets flipped to Yes

$\Rightarrow$ $\overline{HP}$-Solver returns Yes.

Same argument for $ANS(I_{\overline{HP}}) = $ No.

So $\overline{HP} \le_T HP$. $\square$.

Def $\Sigma \neq \phi$, DP A, $L_A \subseteq \Sigma^*$, an **oracle TM M^A** is a TM which can query in any computable manner an oracle for A (A-solver). Given $x \in \Sigma^*$, O^A decides $x \in L_A$ in finite time.

Ex: Given DP A, M^A decides A in "1-step".

$M^A :=$ On input w

Query O^A and return its answer.

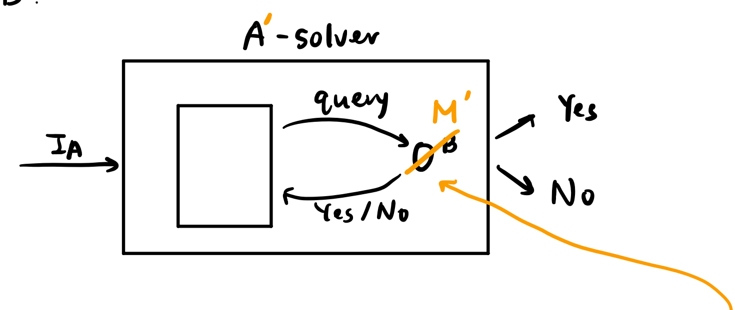
Ex: Let DP ϕ be the decision problem with answer always no. $L_\phi = \phi$.

Given some TM M, oracle TM $\underset{\text{TM } M + O^{\phi}}{\underline{M^{\phi}}}$ is equivalent to M.

Def Given DPs P, Q, we say that **P Turing reduces to Q**, $P \leq_T Q$, if $\exists\ M^{Q}$ that decides P (M^{Q} halts on every input and gives correct Yes / No answer). P is decidable relative to Q.

Theorem If $A \leq_T B$ and B is decidable, then A is decidable.

Proof: $A \leq_T B$.



A'- solver is a TM that decides A.

B decidable $\Rightarrow \exists$ TM M' that decides B. $\square$.

Def DPs A, B, A is **Turing equivalent** to B, $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$.

$\rightarrow$ A and B are at the same level of possibility / impossibility.

Note: $\equiv_T$ is an equiv. relation.

Def Given DP A, an equiv. class of A for $\equiv_T$ is called a **Turing degree**.

$\deg(A) = \{ B : A \equiv_T B \}$.

Partial order on Turing degrees:

$$\underset{a}{\deg(A)} \leq \underset{b}{\deg(B)} \iff A \leq_T B.$$

$$\deg(A) < \deg(B) \iff A \leq_T B \text{ and } \cancel{B \leq_T} A.$$

Def Given DP A, **jump(A)**, **A'**, is the DP s.t.

$L_{A'} = \{ \langle M^A, x \rangle : M^A$ is an oracle TM, M^A halts on $x \}$.

$A = \phi$ $\quad\quad$ $A' = \phi' = HP$

Theorem $\deg(A) < \deg(A')$.

__Proof__: Omit. $\square$

Problem Hierarchy

This partial order creates a chain of strictly "more impossible" classes of problems.

$$0 \ \boxed{\begin{array}{c} \deg(\phi) \\ \{\text{decidable problems}\} \end{array}} \ < \ 0' \ \boxed{\begin{array}{c} \deg(\phi') \\ \{\text{CE probs.}\} \end{array}} \ < \ 0'' \ \boxed{\deg(\phi'')} \ < \ \cdots$$

__Exercise__: Show that any pair of decidable problems are Turing equiv.

Any CE problem has degree at most 0',
but there are non-CE problems with degree
at most 0' as well so 0' != {CE Probs}.