Theory of Computation

Tutorial - Introduction to Grammars

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Plan for today

1. What's a grammar?
2. Grammar formalization
3. Types of grammars
4. Regular grammars
5. Linear grammars
6. Regular grammars and regular languages

What’s a grammar?

What's a grammar?

Grammar formalization

Definition

Definition. A grammar is a 4-tuple $G = (V, T, S, P)$ where

- V is a finite set of variables (upper case symbols)
- T is a finite set of terminals (lower case symbols)
- S is the (unique) start variable
- P is a finite set of productions

Productions in P have the form $x \to y$ where, in their most general form, $x \in (T \cup V)^*$ and $y \in (T \cup V)^*$.

Example

Example. $G = (V = \{S, A, B\}, T = \{a, b\}, S, P)$ where P is

$$S \rightarrow A|\lambda$$
$$A \rightarrow a|bbBa$$
$$B \rightarrow b|\lambda$$

Derivations

Definition. If using a sequence of productions we can obtain y from x we say x **derives** y or $x \Rightarrow^* y$.

Example. From G in the previous example,

$$S \to A|\lambda$$
$$A \to a|bbBa$$
$$B \to b|\lambda$$

the grammar G can generate the string $bbba$

$$S \Rightarrow A \Rightarrow bbBa \Rightarrow bbba \text{ or we can write it as } S \Rightarrow^* bbba$$

Grammars as language generators

Definition. A grammar $G = (V, T, S, P)$ is said to **generate** a language $L(G)$ which is defined as $L(G) = \{w \in T^* : S \Rightarrow^* w\}$.

Example. From G in the previous example,

$$S \rightarrow A|\lambda$$
$$A \rightarrow a|bbBa$$
$$B \rightarrow b|\lambda$$

we have

- $S \Rightarrow \lambda$
- $S \Rightarrow A \Rightarrow a$
- $S \Rightarrow A \Rightarrow bbBa \Rightarrow bba$
- $S \Rightarrow A \Rightarrow bbBa \Rightarrow bbba$

$$L(G) = \{\lambda, a, bba, bbba\}$$

Exercise

Exercise. What is the language generated by the following grammar $G = (\{S, A, B\}, \{a, b\}, S, P)$ where P is

$$S \to Bb|\lambda$$

$$B \to bbB|\lambda$$

Starting with S there are several possible derivations:

- Using the λ derivation: $S \Rightarrow \lambda$
- Using the Bb derivation:
  - $S \Rightarrow Bb \Rightarrow \lambda b = b$
  - $S \Rightarrow Bb \Rightarrow bbBb \Rightarrow bbb$
  - $S \Rightarrow Bb \Rightarrow bbBb \Rightarrow bbbbBb \Rightarrow bbbbb$
  - In general, $S \Rightarrow^* b(bb)^*$

$L(G) = \{\lambda\} \cup \{b^{2n+1} : n \geq 0\}$

Exercise

Exercise. Determine $L(G)$ and *prove* your answer. That is, if you claim that $L(G) = L$, show that this is the case. $G = (\{S\}, \{a, b\}, S, P)$ with P as

$$S \rightarrow aSb|aSbb|\lambda$$

Types of grammars

Types of grammars

The type of a grammar is determined by the form its production rules can take on.

1. Left-linear and right-linear grammars
2. Context-free grammars
3. Context-sensitive grammars
4. Unrestricted grammars

Regular grammars

Left-linear grammars

Definition. A grammar $G = (V, T, S, P)$ is **left-linear** if each of the production rules in P have one of the following forms:

$$A \to Bx$$
$$A \to x$$

Where $A, B \in V$ and $x \in T^*$. Can you give a slightly more formal definition?

Example. The following is a left-linear grammar. What does it generate?

$$S \to Aa|\lambda$$
$$A \to Sb$$

Right-linear grammars

Definition. A grammar $G = (V, T, S, P)$ is **right-linear** if each of the production rules in P have either of the following forms:

$$A \to xB$$
$$A \to x$$

Where $A, B \in V$ and $x \in T^*$.

Example. The following is a right-linear grammar. What does it generate?

$$S \to aA|\lambda$$
$$A \to bS$$

Regular grammars

Definition. A grammar G is **regular** if it is either right-linear or left-linear.

Example. Is the following grammar left-linear? Right-linear? Regular?

$$S \to aaaA|\lambda$$
$$A \to abA|\lambda$$

Example

Example. Consider the grammar $G = \{V = \{S, A\}, T = \{a, b\}, S, P\}$ where P is

$$S \to aS|bS|abA$$
$$A \to aA|bA|\lambda$$

G is a right-linear (and hence a regular) language.

Any string of the form $w_1abw_2, w_1, w_2 \in \{a, b\}^*$ can be generated by G. Suppose $w_1 = ba$, $w_2 = aa$:

$$S \Rightarrow bS \Rightarrow baS \Rightarrow ba\textcolor{red}{ab}A \Rightarrow baabaA \Rightarrow baabaa$$

G generates $L(G) = \{w \in \{a, b\}^* :\ \text{w contains } ab \text{ as a substring }\}$.

Exercise

Exercise. Write a regular grammar that generates
$L = \{w \in \{a, b\}^* : |w| \mod 2 = 0\}$

Linear grammars

Linear grammars

Definition. A grammar $G = (V, T, S, P)$ is linear if each of the production rules have the form $x \to y$ where $x \in V$ and $y \in T^* \cdot (V \cup \{\lambda\}) \cdot T^*$.

Example. The following is a linear grammar. What does it generate?

$$S \to aA|\lambda$$
$$A \to Sb$$

Exercise. Are all linear grammars also regular? Are all regular grammars also linear?

Regular grammars and regular languages

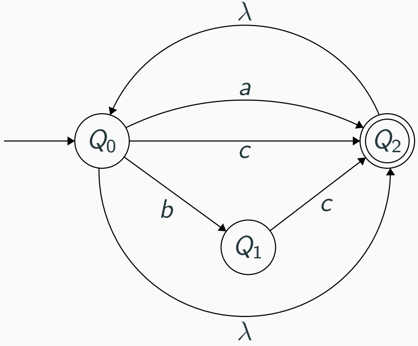
Converting an FA to a right-linear grammar

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an FA (either an NFA or a DFA). We want to create a right-linear grammar $G = (V, T, S, P)$ such that $L(G) = L(M)$. Let $V = Q$, $T = \Sigma$, $S = q_0$. Construct the production rules as follows:

1. For transitions of the form $\delta(q_i, \sigma) = q_j$, where $\sigma \in \Sigma$, create a production rule $q_i \to \sigma q_j$
2. For transitions like $\delta(q_i, \lambda) = q_j$, create a production rule $q_i \to q_j$
3. Create a production rule $q_j \to \lambda$ for $q_j \in S_f$

Exercise

Exercise. Convert the following FA to a right-linear grammar.



Converting a right-linear grammar to an FA

Let $G = (V, T, S, P)$ be a right-linear grammar. We want to create an FA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = L(G)$. Sketch of the procedure:

Exercise

Exercise. Convert the following right-linear grammar to an equivalent FA.

$$
\begin{aligned}
S &\rightarrow aA|a|cC|c|bB|b \\
A &\rightarrow aA|a|bB|b \\
B &\rightarrow bB|b \\
C &\rightarrow cC|c|bB|b
\end{aligned}
$$

Equivalence between regular grammars and FAs

Theorem. Every regular language has a regular grammar that generates it and every regular grammar generates a regular language.[1]

[1] This theorem assumes a result about the relation between right-linear and left-linear grammars. Do you see what it is?