Theory of Computation

Tutorial - Languages

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Plan for today

1. Languages

Languages

Definitions

Definition. An **alphabet** Σ is a finite set of symbols. It must contain at least one symbol.

Example 1. $\Sigma = \{a, b, ..., z\}$

Definition. A **string** is a sequence of symbols from a given alphabet Σ.

Example 2. Given $\Sigma = \{a, b\}$, $w = abba$ is a well-defined string over Σ, but $v = acb$ is not.

Definition. A **language** L is a set of strings defined over an alphabet Σ.

Example 3. Given $\Sigma = \{a, b\}$, $L = \{a, aa, aab\}$ is a finite language.

Example 4. Given $\Sigma = \{a, b\}$, $L = \{a^n : n > 1\}$ is an infinite language.

Operations on strings

Given strings $w = a_1 a_2 \dots a_n$ and $v = b_1 b_2 \dots b_m$ defined over Σ, define the following string operations

String concatenation: $wv = a_1 a_2 \dots a_n b_1 b_2 \dots b_m$

String reversal: $w^R = a_n \dots a_2 a_1$

String length: The number of characters in a string. $|w| = n$ and $|v| = m$

Empty string

Definition. The empty string λ (also ε) is defined as the string with length 0. $|\lambda| = 0$. Equivalent to '' in Java/Python.

Example 1. What is $\lambda^R =$?

Example 2. What is $w\lambda$ for any string w?

More operations on strings

Definition. For every string $v = b_1 b_2 \dots b_m$ defined over Σ,
$v^n = \underbrace{v \cdot v \cdot ... \cdot v}_{\text{n times}}$.

Example 1. What is $v^0 =$?

Example 2. $w = ab, v = ba$. $w^2 v^2 w^0 =$?

Substring

Definition. A string z is a **substring** of a string w if it appears consecutively within w.

Example 1. Let $w = abaa$, what are its possible substrings?

Example 2. Given a string w of length n, how many substrings will it have?

Prefix and suffix

Definition. A string x is a **prefix** of w if there exists a string z such that $w = xz$.

Definition. A string x is a **suffix** of w if there exists a string z such that $w = zx$.

Example 1. Let $w = abbaa$, what are the prefixes and suffixes of w?

Example 2. True or False. For any string w there is exactly one substring x that is both a prefix and a suffix.

Example 3. Let w be a string of length n, how many prefixes will w have?

Operations on languages

Definition. The **union**, **intersection** and **difference** of languages can be applied as set operations.

Example 1. Given $\Sigma = \{a, b\}$, $L_1 = \{a, ab, abab\}$, $L_2 = \{(ab)^n : n \geq 0\}$, what is:

$L_1 \cup L_2 =$

$L_1 \cap L_2 =$

$L_1 - L_2 =$

$L_2 - L_1 =$

Operations on languages - Continued

Definition. The **reverse** of a language is defined as $L^R = \{w^R : w \in L\}$.

Example 2. Given $L_1 = \{a, ab, abab\}$, $L_2 = \{(ab)^n : n \geq 0\}$.

What is $L_1^R =$

What is $L_2^R =$

Operations on languages - Continued

Definition. Given L_1, L_2 the language **concatenation** $L_1 L_2$ is defined as $\{wv : w \in L_1, v \in L_2\}$ (like the cross-product of two sets).

Example 1. Given $L_1 = \{a\}$, $L_2 = \{a^n : n \geq 0\}$, what is:

$L_1 L_2 =$

$L_2 L_1 =$

Example 2. In general, for two languages L_1, L_2, is $L_1 L_2 = L_2 L_1$?

Operations on languages - Continued

Definition. Given a language L, define $L^n = \underbrace{LL \ldots L}_{\text{n times}}$.

Example 1. Let $L_1 = \{a^n b^n : n \geq 0\}$. What is L_1^2?

Example 2. What is $L^0 =$? Give some intuition behind this.

Operations on languages - Continued

Definition. Given a language L, the **star-closure** of L, denoted by L^*, is defined as the following language: $L^* = L^0 \cup L \cup L^2 \cup L^3 \cup \ldots$

Example 1. Given $\Sigma = \{a, b\}$, what is the set of all possible strings over Σ ? .

Example 2. Given $L_4 = \{a, ab\}$ how many strings of length 0, 1 and 2 are there in L_4^*? What are those strings?

Operations on languages - Continued

Defintion. Given a language L, the **positive-closure** is defined as $L^+ = L^1 \cup L^2 \cup L^3 \cup \ldots$

Example 1. Given $L_4 = \{a, ab\}$, what is the shortest string in L_4^+?

Example 2. True or False. $L^+ = L^* - \{\lambda\}$?

Three very important languages

Three languages that we will OFTEN see/use in the proofs/counter-examples: $\emptyset$, $\{\lambda\}$, Σ^*.

Example 1. Given $L = \{a, ab\}$ over $\Sigma = \{a, b\}$. How do we define $\overline{L}$?

Example 2. What is $L\emptyset$?

Example 3. What is $\emptyset^0$?

Example 4. What is $\emptyset^*$?

Exercises

Let $L_1 = \{a^n b^{n+1} : n \geq 0\}$, $L_2 = \{a^n : n \mod 2 = 0\}$ then:

a. $L_1 - L_2 =?$

b. $\Sigma^* \cup L_1^R \overline{L_2}\{\lambda\} =?$

c. What is the shortest string in $L_1^0\{a, b\}\emptyset$?

Exercises - Continued

d. Find languages L_1, L_2 such that $L_1 L_2 = L_1$?

e. Find a language L such that $L^* = L$?

Exercises - Continued

f. True or False. $|L \cup L^2| = |L| + |L^2|$.

g. True or False. $|L \cup L^2| \leq |L| + |L^2|$.