Theory of Computation

Tutorial - Pushdown Automata

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Plan for today

1. What is a Pushdown Automata?
2. Formalization
3. Designing PDAs

What is a Pushdown Automata?

Intuition

Formalization

Definition

Definition.[1] A nondeterministic pushdown automata (npda) is defined by the seven-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where

- Q is a finite set of states
- Σ is the input alphabet; $\Sigma_\lambda = \Sigma \cup \{\lambda\}$
- Γ is the stack alphabet; $\Gamma_\lambda = \Gamma \cup \{\lambda\}$
- δ: $Q \times \Sigma_\lambda \times \Gamma_\lambda \to \mathcal{P}_{\texttt{fin}}(Q \times \Gamma^*)$ is the transition function where $\mathcal{P}_{\texttt{fin}}$ is the finite powerset.[2]
- $q_0 \in \Sigma$ is the start state
- $z \in \Gamma$ is the stack start symbol
- $F \subseteq Q$ is the set of final states

[1] There are many slightly different definitions for PDAs. They are all equivalent in that they all accept exactly the same family of languages.
[2] This wasn't necessary for NFAs because the powerset $\mathcal{P}(Q)$ was always finite.

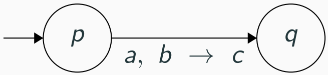
Transition function

Transition rule. The transition

$$\delta(q, a, b) = \{(p, c)\}$$

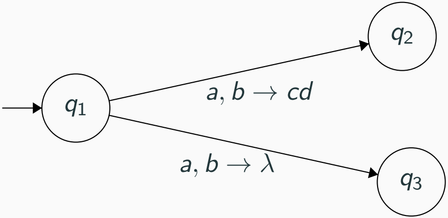
means while reading an $a \in \Sigma_\lambda$ on your input tape and a $b \in \Gamma_\lambda$ at the top of your stack in the state $q \in Q$ you will transition to state $p \in Q$ and will *replace* b at the top of the stack with $c \in \Gamma_\lambda^*$.

This is represented graphically as



Example

Example. If we have the transition rule $\delta(q_1, a, b) = \{(q2, cd), (q3, \lambda)\}$, then this is represented graphically as



- From q_1 to q_2, reading a, popping b from the top of the stack and inserting cd. The leftmost symbol of cd should be at the top of the stack.
- From q_1 to q_3, reading a and removing b from the top of the stack

Instantaneous description

Definition. The **instantaneous description** of a PDA M describes the contents of M at a particular instance in its computation. It is a triple $(q, w, u) \in Q \times \Sigma_\lambda^* \times \Gamma_\lambda^*$ where q is the current state, w is the input string left to read and u is the current state of the stack.

Transition rule. We represent a transition $\delta(q, a, b) = (p, y)$ for the PDA M using instantaneous descriptions as $(q, aw, bx) \succ_M (p, w, yx)$.

Definition. The language accepted by a PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is the set

$$L(M) = \{w \in \Sigma^* : (q_0, w, z) \succ_M^* (p, \lambda, u), p \in F, u \in \Gamma^*\}$$

Example

Example. Consider the following PDA M. Suppose \$ is the stack start symbol and assume it is already place onto the stack. Let's rune through some computations.

Example

Input String: aabbb

- Input: aabbb
- Stack: |\$
- State: q_0

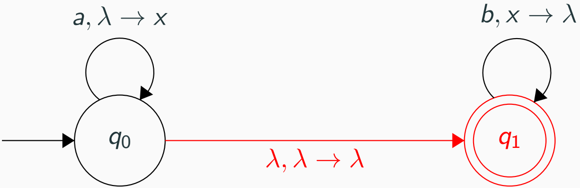
Example

Input String: aabbb

- Input: aabbb
- Stack: $x|\$$
- State: q_0

Example

Input String: aabbb



- Input: aabbb
- Stack: $xx|\$$
- State: q_0

Example

Input String: aabbb

- Input: aabbb
- Stack: $xx|\$$
- State: q_1

Example

Input String: aabbb

- Input: aabbb
- Stack: $x|\$$
- State: q_1

Example

Input String: aabbb

- Input: aabbb
- Stack: |\$
- State: q_1

STUCK! Rejected.

Example

What is the language accepted by this PDA?

Exercise

Exercise. What symbols should be in the stack after the following transitions?

- $\delta(q_1, a, \lambda) = \{(q_2, cbb)\}$, Stack: $b|\$$.
- $\delta(q_1, a, c) = \{(q_2, \lambda)\}$, Stack: $cb|\$$.
- $\delta(q_1, a, \lambda) = \{(q_2, \lambda)\}$, Stack: $cb|\$$.
- $\delta(q_1, a, \$) = \{(q_2, \lambda)\}$, Stack: $|\$$.

Designing PDAs

Tips

- Use the same pattern matching abilities as DFAs/NFAs,
- But now augment this with a stack which can count the occurences of these patterns.
- Caveat: You can only use 1 stack and once you read/pop, you can't 'go back'.[3]

[3] This intuitive limitation will be discussed later when we show examples of languages that can't be accepted by PDAs.

Exercise

Exercise. Give a pushdown automata that accepts the language $L = \{a^i b^j c^k | i, j, k \geq 0 \text{ and } i + j = k\}$

Exercise

Exercise. Give a pushdown automata that accepts the language
$L = \{a^{2n}b^{3n} : n \geq 0\}$

Exercise

Exercise. Give a pushdown automata that accepts the language $\{w \in \{a, b, c\}^* : n_a(w) + n_b(w) \neq n_c(w)\}$